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Mathematical Aspects of the Quantum Theory of Fields

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Part V. Fields Modified by Linear Homogeneous Forces

22. *Boson Fields under the Influence of Spring Forces*

In Part III we considered a boson field caused by the presence of a source distribution. We found that the source distribution contributes a right member to the differential equations for the field quantity $\Xi(x, t)$, making the equations non-homogeneous. In the present part we shall discuss the modifications of a field caused by forces which are linear in the field quantity Ξ; such forces will simply be called "spring forces." It is characteristic for this type of modification that the differential equation for the field quantity remains linear and homogeneous. The main result of this part is that the transformations needed to describe such modifications can be determined completely; in fact, they can be expressed explicitly in terms of the solution of the corresponding unquantized problem.

In the first sections of the present part we shall be concerned with boson fields, but in Section 28 we shall discuss similar problems for fermion fields, specifically, for fields of electrons and positrons endowed with Dirac's energy. Particular attention will be given to the problem of "vacuum polarization".

The boson field $\mathfrak{F}$ to be considered in the present section will be described by a field quantity or potential Ξ which is a function $\Xi(x, t)$ of the position x and the time t and satisfies the differential equation

$$(\nabla_t^2 - \nabla_x^2 + \mu^2)\Xi + Q^s\Xi = 0. \tag{22.1}$$

The symbol Q^s signifies a functional operator which acts on functions of x; it will be referred to as the "disturbance operator". The term $-Q^s\,\Xi(x, t)$ represents the force per unit mass exerted on the field at the point x at the time t.

The initial conditions for the solution $\Xi(x, t)$ of equation (22.1) are the same as those for the undisturbed field, cf. (2.1),

$$\begin{aligned} [\nabla_t\Xi(x', 0), \Xi(x'', 0)] &= -i\delta(x' - x''), \\ [\Xi(x', 0), \Xi(x'', 0)] &= 0, \\ [\nabla_t\Xi(x', 0), \nabla_t\Xi(x'', 0)] &= 0. \end{aligned} \tag{22.2}$$

The field $\mathfrak{F}$ may be an electromagnetic or a neutral meson field. However, it may also be an elastic medium provided the density and the elastic constants involved are taken equal to unity. In the following we usually refer to this interpretation. In an elastic medium, the quantity Ξ stands for the displacement or elongation. The position variable x may range either over the whole or a part of three-, two-, or one-dimensional space; if its range is only a part of one of these spaces, appropriate boundary conditions must be imposed on the functions of x on which the operator ∇_x^2 acts.

Adjoint and Conjugate Operators

Before formulating conditions on the disturbance operator Q we note that the notions of "real" adjointness and "real" symmetry will often be used instead of Hermitian adjointness and symmetry; also we shall assign to each operator a "complex conjugate". These notions are defined only with respect to a functional representation; we shall use the x-representation for this purpose.

The *complex conjugate* $\bar{\mathrm{A}}$ of an operator A is defined by the condition that its x-representer $\bar{\mathrm{A}}^x$ transforms a function $\psi(x)$ into the complex conjugate of the function $\mathrm{A}^x\,\overline{\psi(x)}$,

$$\bar{\mathrm{A}}^x\psi(x) = \overline{\mathrm{A}^x\overline{\psi(x)}}. \tag{22.3}$$

Here we assume that A^x is applicable to $\overline{\psi(x)}$ if it is applicable to $\psi(x)$. The *real adjoint* will be defined for operators A which have a Hermitian adjoint. Using the notation[1] $'\mathrm{A}$ for the real adjoint and A^* for the Hermitian adjoint we define

$$'\mathrm{A} = \bar{\mathrm{A}}^*. \tag{22.4}$$

Accordingly,

$$\int {'\mathrm{A}^x}\psi(x)\cdot\psi^{(1)}(x)\,dx = \int \psi(x)\mathrm{A}^x\psi^{(1)}(x)\,dx \tag{22.5}$$

whenever A^x is applicable to $\psi(x)$, $\overline{\psi(x)}$, $\psi^{(1)}(x)$. If $'\mathrm{A} = \mathrm{A}$, we say that A is *real symmetric*. If $\bar{\mathrm{A}} = \mathrm{A}$ and $'\mathrm{A} = \mathrm{A}$, and hence $^*\mathrm{A} = \mathrm{A}$, we say that A is *real and symmetric*.

We now require that the disturbance operator Q be real and symmetric. More specific conditions on the nature of Q will be formulated later, see Section 26; but at present we may visualize Q^x as an integral operator, with a real and symmetric kernel $q(x, x')$,

$$Q^x\psi(x) = \int q(x, x')\psi(x')\,dx'. \tag{22.6}$$

The force expressed by the term $-Q^x\psi(x)$ may be interpreted as the force due to a distribution of springs, connecting the points x' with the points x, the kernel $q(x, x')$ being the density of the spring constant.

[1] Our notations differ from those customary in work on quantum physics.

Various Types of Problems

A number of generalizations of this problem may be treated in the same manner. These generalizations result from modification of the independent variable x, the differential dx, and the operator $\mu^2 - \nabla_x^2$. These modifications could in fact be subsumed under our theory by re-interpreting the terms x, dx, $\mu^2 - \nabla_x^2$ in an appropriate way.

One particularly interesting generalization is concerned with the interaction between several media; for example, between two two-dimensional membranes or two one-dimensional strings. In order to include such cases in our theory it is necessary to introduce instead of x a more general independent variable; we shall use the pair (x, ρ) in which ρ is a label running over the values $1, \cdots, r$ if r fields are considered. The integral $\int |\psi(x)|^2 dx$ is to be replaced by

$$\sum_{\rho=1}^{r} \int^{\rho} |\psi(x, \rho)|^2 dx.$$

Both the domain of integration and the rest mass μ may be different for each field. The term μ in (22.1) is accordingly to be replaced by μ^ρ and the operator Q^x by $Q^{\rho,\rho}$; this operator then incorporates the interaction between the various media.

Finally, cases in which masses are concentrated on two-, or one-dimensional manifolds or at single points may also be subsumed under the theory. In such cases the differential $dm(x)$ of an appropriate measure function must be substituted for dx and an appropriate undisturbed energy operator must be adopted in place of $\mu^2 - \nabla_x^2$.

For example, we may consider two "media", one extending over the whole space, the other concentrated at the point $x = 0$. The field quantity $\Xi(x, \rho)$ is then defined for all values of x when $\rho = 1$ but only for $x = 0$ if $\rho = 2$. Instead of $\Xi(x, \rho)$ we may use a pair $\{\Xi(x), \Xi_0\}$ to describe the field, letting $\Xi(x) = \Xi(x, 1)$ refer to $\rho = 1$ and $\Xi_0 = \Xi(0, 2)$ refer to $\rho = 2$. As the unit form for any pair $\{\psi(x), \psi_0\}$ we may choose

$$\int |\psi(x)|^2 dx + |\psi_0|^2.$$

The mass concentrated at the point $x = 0$ will be denoted by m but we assume that the rest mass of the extended field vanishes so that $(\Omega^x)^2 = -\nabla_x^2$ for this field. The corresponding term for the point mass is assumed to be of the form $\Omega_0^2 = c_0$ where c_0 is the constant of the spring which connects this mass with a support.

We suppose that the interaction of the extended field with the single mass is provided by springs with the density $q(x)$. Therefore, the interaction operator $Q^{\rho,\rho}$ transforms any pair $\{\psi(x), \psi_0\}$ into the pair

$$Q^{\rho,\rho}\{\psi(x), \psi_0\} = \left\{q(x)\psi_0 , \int q(x)\psi(x)\, dx\right\}.$$

Evidently, this operator is real and symmetric.

Instead of (22.1) the differential equation becomes

$$(\nabla_t^2 - \nabla_x^2)\Xi + q(x)\Xi_0 = 0,$$

$$m\nabla_t^2\Xi_0 + \int q(x)\Xi(x)\,dx = 0;$$

and instead of (22.2) the initial conditions are

$$\nabla_t\Xi(x', 0)\cdot\Xi(x'', 0) - \Xi(x'', 0)\nabla_t\Xi(x', 0) = -i\delta(x' - x''),$$

$$m\nabla_t\Xi_0\cdot\Xi_0 - \Xi_0 m\nabla_t\Xi_0 = -i.$$

A special case of this problem is treated in the thesis of W. Sollfrey [50], namely the case in which the mass interacts with the medium only through a single spring attached to the point $x = 0$. Here

$$q(x) = c_1\delta(x),$$

if c_1 is the constant of the single spring. This point interaction makes the operator Q^* so singular that our treatment is not applicable. On the other hand, our treatment will explain the occurrence of infinities found by Sollfrey.

Modified and Unmodified Particle Representation

Instead of describing the field by means of the quantity $\Xi(x, t)$ we may describe it in terms of particles. With the aid of the energy operator

(22.7) $$\Omega^* = [\mu^2 - \nabla_x^2]^{1/2}$$

we introduce the creation and annihilation operators

(22.8) $$A^{\mp}(x, t) = (\Omega^*/2)^{1/2}\Xi(x, t) \mp i(2\Omega^*)^{-1/2}\nabla_t\Xi(x, t),$$

the number operator

(22.9) $$N(t) = \int A^+(x, t)A^-(x, t)\,dx,$$

and the energy operator

(22.10) $$H_A(t) = \int A^+(x, t)\Omega^* A^-(x, t)\,dx,$$

see (9.5), (10.8), (10.9).

In order to justify the designation of $A^{\mp}(x, t)$ as creation and annihilation operators it is necessary to show that they satisfy the appropriate commutation laws and, furthermore, that a particle representation

(22.11) $$\Phi \underset{N(t)}{\leftrightarrow} \{\psi_n(s)_n(t)\}$$

of the states of the field is associated with them.

The main object of our investigation is to determine the transition probability, i.e. the probability of finding n particles at the time t, or as $t \to \infty$, if no particles were present at the time $t = 0$.

Neither the number operator N, nor the energy operator H_A is constant in time. However, the sum of the energy operator $H_A(t)$ and the interaction operator $V(t)$, the total energy operator

$$H_{tot} = H_A(t) + V(t), \tag{22.12}$$

is constant. The interaction operator $V(t)$ is given by

$$\begin{aligned} V(t) &= \frac{1}{2} \int \Xi(x, t) Q^x \Xi(x, t)\, dx \\ &= \frac{1}{4} \int [A^+(x, t) + A^-(x, t)](\Omega^x)^{-1/2} Q^x (\Omega^x)^{-1/2} [A^+(x, t) + A^-(x, t)]\, dx. \end{aligned} \tag{22.13}$$

The constancy of the operator H_{tot} is easily verified formally from the differential equations (21.1).

Whether or not the operator $V(t)$ is defined depends on the nature of the operator Q. It could be shown that $V(t)$ can be defined if $\Omega^{-1/2}\, Q\, \Omega^{-1/2}$ is an integral operator with a quadratically integrable kernel. Since we shall not actually work with the operator $V(t)$ no details will be given.

Modified Energy Operator for Single Particles

It is our aim to introduce a "modified" particle representation of the states of the field such that H_{tot} is the associated energy operator except for an additive constant.

The modified particles are endowed with an energy operator which differs from the energy operator Ω^x associated with the unmodified particles. The simplest way to determine the proper expression of the modified particle energy operator is to compare the unmodified and modified field energy operators in the improper forms

$$\frac{1}{2} \int [\nabla_x \Xi^2 + \Xi(\mu^2 - \nabla_x^2)\Xi]\, dx$$

and

$$\frac{1}{2} \int [(\nabla_x \Xi)^2 + \Xi(\mu^2 - \nabla_x^2)\Xi + \Xi Q^x \Xi]\, dx.$$

Clearly, the second expression is of the same form as the first, but we have

$$\Upsilon^x = (\mu^2 - \nabla_x^2 + Q^x)^{1/2} \tag{22.15}$$

in place of $\Omega^x = (\mu^2 - \nabla_x^2)^{1/2}$, see (22.7). This operator Υ is defined if the operator $\mu^2 - \nabla_x^2 + Q^x$ is non-negative; if it is positive definite, the inverse Υ^{-1} of Υ is defined. We assume that the disturbance Q is such that this is the case.

We suppose that the spectral representation of the operator Υ is "known". We may then consider all functions of Ω and Υ as known. At the beginning of this part we stated that our problem can be solved explicitly; this statement can now be made specific: we shall solve our problem explicitly in terms of the operators Ω and Υ.

Employing the operator Υ we can interpret our problem with reference to the single particles of which we may consider the field to be composed. Instead of saying that forces are exerted on the field we may say that forces act on the single particles. Instead of describing this force explicitly it is sufficient to describe the potential energy associated with it. The operator which corresponds to this potential energy is simply the difference $\Upsilon - \Omega$. We may also say that the disturbance is such that Υ has replaced Ω as energy operator.

Modified Creation and Annihilation Operators

In terms of the modified energy operator Υ we may introduce "modified" creation and annihilation operators by

$$B^{\pm}(x, t) = (\Upsilon^s/2)^{1/2}\Xi(x, t) \mp i(2\Upsilon^s)^{-1/2}\nabla_t\Xi(x, t); \tag{22.15}$$

the associated "modified" number operator is

$$M = \int B^{+}(x, t)B^{-}(x, t)\, dx \tag{22.16}$$

and the "modified" energy operator is

$$H_B = \int B^{+}(x, t)\Upsilon^s B^{-}(x, t)\, dx. \tag{22.17}$$

The differential equation (22.1) for the operators Ξ leads to the equation

$$\nabla_t B^{\pm}(x, t) = \pm i\Upsilon^s B^{\pm}(x, t) \tag{22.18}$$

for the operators $B^{\pm}$. Evidently the solution is

$$B^{\pm}(x, t) = \exp\{\pm it\Upsilon^s\}B^{\pm}(x, 0), \tag{22.19}$$

provided that the initial values $B^{\pm}(x, 0)$ are given.

Of course, it is assumed that these initial values obey the commutation laws; it then follows that the $B^{\pm}(x, t)$ also obey these laws.

The constancy of M and H_B is an immediate consequence of (22.19).

We may now express the solution $\Xi(x, t)$ of equation (22.1) in terms of the initial data $\Xi(x, 0)$, $\nabla_t\Xi(x, 0)$ by inverting relation (22.15) on the assumption that this relation holds initially. It can then be verified that $\Xi(x, t)$ and $\nabla_t\Xi(x, t)$ obey the commutation laws for each value of t.

With the aid of (22.8) we can finally express the operators $A^{\pm}(x, t)$ in terms of their initial values. One then easily verifies that these operators obey the commutation laws.

It is necessary to express the operators $A^{\pm}$ directly in terms of the operators $B^{\pm}$. Eliminating Ξ and $\nabla_t \Xi$ from relations (22.15) and (22.9) we find first

$$B^{+}(x, t) + B^{-}(x, t) = (\Upsilon^{1/2}\Omega^{-1/2})^{x}[A^{+}(x, t) + A^{-}(x, t)],$$

(22.20)

$$B^{+}(x, t) - B^{-}(x, t) = (\Upsilon^{-1/2}\Omega^{1/2})^{x}[A^{+}(x, t) - A^{-}(x, t)],$$

and consequently,

(22.21) $$B^{\pm}(x, t) = Y_{+}^{x}A^{\pm}(x, t) + Y_{-}^{x}A^{\mp}(x, t),$$

if we introduce the operators $Y_{\pm}$ by

(22.22) $$2Y_{\pm} = \Upsilon^{1/2}\Omega^{-1/2} \pm \Upsilon^{-1/2}\Omega^{1/2}.$$

Introducing the operators $Z_{\pm}$ by

(22.23) $$2Z_{\pm} = \Omega^{1/2}\Upsilon^{-1/2} \pm \Omega^{-1/2}\Upsilon^{1/2},$$

we have inversely

(22.24) $$A^{\pm}(x, t) = Z_{+}^{x}B^{\pm}(x, t) + Z_{-}^{x}B^{\mp}(x, t).$$

Note that the modified and unmodified creation and annihilation operators are related by linear homogeneous relations; this will be the starting point of our procedure in determining the modified particle representation.

Remarks About Energy Operators

At present we add a few remarks concerning the relationship between the energy operators H_B and H_{tot}, see (22.17) and (22.12). Inserting the expressions (22.24) into (22.12) and using the commutation laws we find

(22.25) $$H_{tot} = H_B + h.$$

The term h is an integral which can be written in the form

$$h = \frac{1}{4}\int [A^{-}(x)J^{x}A^{+}(x) - A^{+}(x)J^{x}A^{-}(x)]\, dx$$

when the operator J is defined by

$$J = J_{+} + J_{-},$$

$$J_{\pm} = \Omega^{\mp 1/2}[\Upsilon - \Omega]\Omega^{\mp 1/2}.$$

Suppose now the operators $J_{\pm}^{x}$ are integral operators with kernels $j_{\pm}(x', x'')$ which possess finite traces

$$Tr\, J_{\pm} = \int j_{\pm}(x, x)\, dx.$$

Then we find

$$h = \tfrac{1}{4}\, Tr\, J_{+} + \tfrac{1}{4}\, Tr\, J_{-}.$$

We may give this expression a different form by using the fact that $Tr\, AB = Tr\, BA$ for any two operators. Supposing that the operator $\Upsilon - \Omega$ possesses a trace we have

$$Tr\, J_{-} = Tr\, \Omega^{-1/2}(\Omega^{-1/2}J)$$
$$= Tr\, (\Omega^{-1/2}J)\Omega^{-1/2} = Tr\, (\Upsilon - \Omega);$$

hence

$$h = \tfrac{1}{2}\, Tr\, (\Upsilon - \Omega). \tag{22.26}$$

The condition that this trace must be finite involves a severe restriction on the nature of the operator $Q = \Upsilon^2 - \Omega^2$. Closely related conditions on the disturbance Q will be formulated in Section 26.

Modified Particle Representation

Our main objective is to determine the particle representation (22.13)

$$\Phi \underset{N(t)}{\leftrightarrow} \{\psi_n(x)_n(t)\} \tag{22.27}$$

associated with the operators $A^{-}(x, t)$ if the representation of the state is given for the initial time $t = 0$. To this end we use the modified particle representation

$$\Phi \underset{M(t)}{\leftrightarrow} \{\chi_m(x_m)(t)\}.$$

In analogy with the procedure described in Part III, cf. (14.10), we determine a transformation T_M^N and its inverse T_N^M which transform the functions $\psi_m(x)_m(t)$ and $\chi_m(x)_m(t)$ into each other,

$$\chi_m(s)_m(t) = T_M^N \psi_n(s)_n(t),$$
$$\psi_n(s)_n(t) = T_N^M \chi_m(s)_m(t). \tag{22.28}$$

Suppose the transformations T_N^M, T_M^N are known. Then we may obtain $\chi_m(x)_m(0)$ from

$$\chi_m(x)_m(0) = T_M^N \psi_n(x)_n(0);$$

furthermore,

$$\chi_m(x)_m(t) = \exp\{-itH_B\}\chi_m(x)_m(0)$$

from (22.19), and finally

$$\psi_n(x)_n(t) = T_N^M \chi_m(x)_m(t).$$

Combining these formulas we find

$$\psi_n(x)_n(t) = T_N^M \exp\{-itH_B\} T_M^N \psi_n(x)_n(0). \tag{22.29}$$

The transition probabilities can then be calculated.

The transformation (22.8) will also be determined directly, without reference to modified particles as intermediaries.

Canonical Transformations

As in Part III we shall derive the transformation T with the aid of a canonical transformation, see Section 14. We introduce the "trivial" transformation I^M_N which transforms the M-representers $\chi_m(x)_m$ of any state Φ into the N-representers

$$\chi_n(x)_n = I^M_N \chi_m(x)_m$$

of that state Φ' whose N-representers happen to be the same functions as the M-representers of Φ, cf. (14.17). We wish to find the unitary transformation T which transforms the state Φ into the states Φ',

$$T\Phi = \Phi'. \tag{22.30}$$

Let T^N be the N-representer of the operator T; then the M-representers of the state $\Phi = T^{-1}\,\Phi'$ are given by

$$\chi_m(x)_m = I^N_M T^N \psi_n(x)_n \tag{22.31}$$

in terms of the N-representers of Φ. In other words, the desired transformation of the N-representers into the M-representers of a state Φ is given by

$$T^N_M = I^N_M T^N, \tag{22.32}$$

cf. (14.22). Our problem has thus been reduced to the determination of the unitary operator T.

The transformation of the modified vacuum state Φ^B_{vac} into the modified vacuum state Φ^A_{vac} is of particular interest. Since the N-representers of Φ^A_{vac} are the same as the M-representers of Φ^B_{vac}, namely, $\{1, 0, 0, \cdots\}$, it is clear from (22.30) that these states are related by

$$\Phi^A_{vac} = T\Phi^B_{vac}. \tag{22.33}$$

In fact, the knowledge of the transformation of Φ^B_{vac} into Φ^A_{vac} expressed in terms of the creation operators B^+ would already enable one to construct the general transformation of M- into N-representers. It is, however, much simpler to give the complete transformation T directly.

As in Section 14, we shall characterize the operator T, without reference to representers, by the condition

$$A^*(x')T = TB^*(x') \tag{22.34}$$

cf. (14.21). We shall determine the unitary operator T so that it satisfies this condition and define the representers $\chi_m(x)_m$ of a state Φ by (22.31). It is clear that the operators $B^*(x)$ are exactly the creation and annihilation operators associated with the representation of Φ in terms of these functions $\chi_m(x)_m$.

23. *General Homogenous Linear Transformation of Creation and Annihilation Operators*

Our procedure for determining a unitary operator T which satisfies relation (22.35) will be developed quite generally under the assumption that the operators B^{*} and A^{*} are connected by any homogeneous linear transformation which guarantees that the operators B^{+} and B^{-} are conjugate to each other and obey the commutation laws provided the operators A^{+} and A^{-} have these properties. Relation (22.21) is just a special case of such a transformation. In fact, we shall develop our procedure simultaneously for fermion and boson annihilation and creation operators.

We shall also adopt a general quantum variable s with a measure differential $dm(s)$ instead of x and dx.

It is convenient to consider the pair of operators $A^{*}(s)$ as one entity

(23.1) $$\mathcal{A}(s) = \{A^{-}(s), A^{+}(s)\};$$

more generally, to a pair of functions $v^{*}(s)$ in $\mathfrak{H}$, i.e., to a pair of quadratically integrable functions, we assign the "pseudo-function"

(23.2) $$v(s) = \{v^{-}(s), v^{+}(s)\}.$$

To two such pseudo-functions, $v(s)$ and $w(s)$, we assign the "pseudo-product"

(23.3) $$w \circ v = \int [w^{+}(s)v^{-}(s) \mp w^{-}(s)v^{+}(s)]\, dm(s).$$

Henceforth the upper sign is to be used if the operators A^{*} refer to a boson field, the lower sign if the A^{*} refer to a fermion field. Evidently,

(23.3)′ $$v \circ w = \mp(w \circ v)$$

We also use commutators and anticommutators

(23.4) $$[\Lambda_1 ; \Lambda_2] = \Lambda_1\Lambda_2 \mp \Lambda_2\Lambda_1$$

in connection with boson and fermion fields respectively (the semicolon is used in order to make it unnecessary to indicate the anticommutator by a special subscript).

With the aid of functions v^{*} and w^{*} belonging to appropriate subspaces of $\mathfrak{H}$ we may form the field operators[2] $w \circ \mathcal{A}$ and $\mathcal{A} \circ v$; c.f. (8.5), (8.13). One readily verifies that the commutation laws (8.16), (8.20) and (8.16)*, (8.20)* are equivalent with the validity of the relation

(23.5) $$[w \circ \mathcal{A}; \mathcal{A} \circ v] = w \circ v$$

or

(23.5)′ $$[\mathcal{A} \circ w; v \circ \mathcal{A}] = w \circ v$$

for arbitrary pseudo-functions v and w.

[2] A field operator acts on states of the field. We shall use this term whenever it is desirable to distinguish such operators from the "operators" which act on states of single particles.

With the aid of four bounded operators $L^S_{\mp\mp}$ acting on functions in $\mathfrak{H}$ we form the "pseudo-operator"

$$\mathcal{L}^S = \begin{pmatrix} L^S_{--} & L^S_{-+} \\ L^S_{+-} & L^S_{++} \end{pmatrix} \tag{23.6}$$

which transforms the pseudo-function $v(s)$ into the pseudo-function

$$\mathcal{L}^S v(s) = \{L^S_{--} v^-(s) + L^S_{-+} v^+(s),\ L^S_{+-} v^-(s) + L^S_{++} v^+(s)\}. \tag{23.7}$$

We write the pseudo-product of $w(s)$ and $\mathcal{L}^S v(s)$ as $w \circ \mathcal{L}\, v$, omitting the superscript S; in fact, we shall omit this superscript whenever possible without inconsistency. To a pseudo-operator $\mathcal{L}$ we assign its "pseudo-adjoint" $'\mathcal{L}$ by the condition

$$\mathcal{L}w \circ v = w \circ {}'\mathcal{L}v. \tag{23.8}$$

Evidently,

$$'\mathcal{L}^S = \begin{pmatrix} 'L^S_{++} & \mp 'L^S_{-+} \\ \mp 'L^S_{+-} & 'L^S_{--} \end{pmatrix}. \tag{23.9}$$

Here the operators $'L^S_{\mp\mp}$ are the real adjoints of the operators $L^S_{\mp\mp}$. To appropriate functions $f(\lambda)$ we may assign the pseudo-operators $f(\mathcal{L})$; clearly

$$'f(\mathcal{L}) = f('\mathcal{L}). \tag{23.10}$$

To the pair of pseudo-operators $\mathcal{L}$, $\mathfrak{M}$ we may assign the commutator $[\mathcal{L}, \mathfrak{M}]$. Note that the symbol [,] indicates the proper commutator

$$[\mathcal{L}, \mathfrak{M}] = \mathcal{L}\mathfrak{M} - \mathfrak{M}\mathcal{L}. \tag{23.11}$$

Evidently,

$$'[\mathcal{L}, \mathfrak{M}] = -['\mathcal{L}, '\mathfrak{M}]. \tag{23.12}$$

A pseudo-operator $\mathcal{P}$ will be called "pseudo-antisymmetric" if

$$'\mathcal{P} = -\mathcal{P}, \tag{23.13}$$

i.e. if

$$'P_{++} = -P_{--}, \qquad 'P_{--} = -P_{++}, \tag{23.14$_1$}$$

$$'P_{+-} = \pm P_{+-}, \qquad 'P_{-+} = \pm P_{-+}. \tag{23.14$_2$}$$

One important fact about pseudo-antisymmetric operators is that *the commutator* $[\mathcal{P}_1, \mathcal{P}_2]$ *of two such operators is also pseudo-antisymmetric.* From relation (23.10) it can be inferred, moreover, that the matrix $\exp \mathcal{P}$ has the property

$$'(\exp \mathcal{P}) = (\exp \mathcal{P})^{-1} \tag{23.15}$$

if $\mathcal{P}$ is pseudo-antisymmetric.

We shall frequently impose an additional condition on the pseudo-operators $\mathcal{L}$, namely

$$\overline{L}_{\sigma,\tau} = L_{-\sigma,-\tau}\,. \tag{23.16}$$

Here σ and τ stand for the labels $+$ or $-$, and $\overline{L}$ is the complex conjugate of L. Pseudo-operators $\mathcal{L}$ whose components satisfy condition (23.16) will be called "pseudo-hermitian." If a pseudo-operator $\mathcal{L}$ is speudo-hermitian every real function of $\mathcal{L}$ is also pseudo-hermitian.

We now assume that a linear transformation of the "unmodified" annihilation and creation operators $\mathcal{A} = \{A^-(x), A^+(x)\}$ into modified operators $\mathcal{B} = \{B^-(x), B^+(x)\}$ of the form

$$\mathcal{B} = \mathcal{Y}\mathcal{A} \tag{23.17}$$

is given, such that the pseudo-operator $\mathcal{Y}$ is pseudo-hermitian and satisfies the condition

$$\mathcal{Y}'\mathcal{Y} = '\mathcal{Y}\mathcal{Y} = 1 \tag{23.18}$$

or

$$'\mathcal{Y} = \mathcal{Y}^{-1}. \tag{23.18$'$}$$

By assumption, the hermitian adjoint of the operator A^σ is $(A^\sigma)^* = A^{-\sigma}$. Hence the hermitian adjoint of the operator $B^\sigma = Y_{\sigma,-}A^- + Y_{\sigma,+}A^+$ is $(B^\sigma)^* = \overline{Y}_{\sigma,-}A^+ + \overline{Y}_{\sigma,+}A^- = Y_{-\sigma,-}A^- + Y_{-\sigma,+}A^+ = B^{-\sigma}$. Thus the pseudo-hermitian character of $\mathcal{Y}$ insures that *the operators $B^\sigma(x)$ are hermitian adjoints of each other.*

We also maintain that condition (23.18), imposed on the pseudo-operator $\mathcal{Y}$, insures that *the operators $B^\sigma(x)$ satisfy the commutation laws if*, as assumed, the operators $A^\sigma(x)$ do. In fact, from (23.8), (23.5), and (23.18) we deduce

$$[w \circ \mathcal{B}; \mathcal{B} \circ v] = [{}'\mathcal{Y}w \circ \mathcal{A}; \mathcal{A} \circ {}'\mathcal{Y}v] = {}'\mathcal{Y}w \circ {}'\mathcal{Y}v = w \circ \mathcal{Y}'\mathcal{Y}v$$

and therefore relation $[w \circ \mathcal{B}; \mathcal{B} \circ v] = w \circ v$; which is equivalent to the commutation laws.

Suppose a pseudo-antisymmetric pseudo-hermitian operator $\mathcal{R}$ exists such that

$$\mathcal{Y} = \exp \mathcal{R}. \tag{23.19}$$

Then $\mathcal{Y}$ is also pseudo-hermitian and we conclude that in relation (23.15) the operator $\mathcal{Y} = \exp \mathcal{R}$ satisfies the condition $\mathcal{Y}'\mathcal{Y} = 1$, cf. (23.18). In the following we assume that $\mathcal{Y}$ is of this form.

The canonical transformation T will first be described in terms of the pseudo-operator $\mathcal{R}$. Eventually, however, we shall describe this transformation in different ways independently of the assumption that the pseudo-operator $\mathcal{R}$ exists. Incidentally, for fermion fields the existence of the pseudo-operator $\mathcal{R}$ is always certain since in this case the pseudo-operator $\mathcal{Y}$ is unitary.

The transformation (23.15) of operators A^σ into operators B^σ, connected with the modification of a boson field under the influence of spring forces, is of the form (23.17) with a pseudo-operator

(23.20) $$\mathcal{Y} = \begin{pmatrix} Y_+ & Y_- \\ Y_- & Y_+ \end{pmatrix}$$

whose components are given by

(23.21) $$\begin{aligned} Y_{--} = Y_{++} = Y_+ &= \tfrac{1}{2}[\Upsilon^{1/2}\Omega^{-1/2} + \Upsilon^{-1/2}\Omega^{1/2}], \\ Y_{+-} = Y_{-+} = Y_- &= \tfrac{1}{2}[\Upsilon^{1/2}\Omega^{-1/2} - \Upsilon^{-1/2}\Omega^{1/2}], \end{aligned}$$

according to (23.16). Clearly, this pseudo-operator $\mathcal{Y}$ satisfies condition (23.16) since Υ and Ω are real. Also, the pseudo-adjoint $'\mathcal{Y}$ of $\mathcal{Y}$ is given by

(23.20)′ $$'\mathcal{Y} = \begin{pmatrix} Z_+ & Z_- \\ Z_- & Z_+ \end{pmatrix}$$

with

(23.21)′ $$\begin{aligned} Z_+ &= \tfrac{1}{2}[\Omega^{-1/2}\Upsilon^{1/2} + \Omega^{1/2}\Upsilon^{-1/2}] \\ Z_- &= \tfrac{1}{2}[\Omega^{-1/2}\Upsilon^{1/2} - \Omega^{1/2}\Upsilon^{-1/2}]. \end{aligned}$$

As seen from (22.23), the operator $'\mathcal{Y}$ is the inverse of $\mathcal{Y}$, so that condition (23.18)′ is satisfied. It is not obvious, though, whether or not this pseudo-operator $\mathcal{Y}$ is of the form $\mathcal{Y} = \exp \mathcal{R}$. In constructing the canonical transformation T we shall at first assume that it is, but this assumption will be eventually discarded; for, we shall be able to verify the correctness of the final expression of T independently.

Pseudo-biquantized Operators

Our procedure for finding a canonical transformation T explicitly makes extensive use of certain field operators $[\mathcal{P}]$ which are assigned to pseudo-antisymmetric pseudo-operators $\mathcal{P}$ by the formula

(23.22) $$\begin{aligned} [\mathcal{P}] &= \tfrac{1}{2}\mathfrak{a} \circ \mathcal{P}\mathfrak{a} \\ &= \tfrac{1}{2}A^+P_{--}A^- + \tfrac{1}{2}A^+P_{-+}A^+ \mp \tfrac{1}{2}A^-P_{+-}A^- \mp \tfrac{1}{2}A^-P_{++}A^+ \\ &= \tfrac{1}{2}\int A^+(s)P^s_{--}A^-(s)\,dm(s) + \tfrac{1}{2}\int A^+(s)P^s_{-+}A^+(s)\,dm(s) \\ &\quad \mp \tfrac{1}{2}\int A^-(s)P^s_{+-}A^-(s)\,dm(s) \mp \tfrac{1}{2}\int A^-(s)P^s_{++}A^+(s)\,dm(s). \end{aligned}$$

This assignment, which is a basic tool in the following investigations, will be studied in detail. It is analogous to the assignment of the biquantized operator

A^+PA^- to the single particle operator P. We shall, therefore, call the operator $[\mathcal{P}]$ "pseudo-biquantized". For this assignment we are using the same bracket notation we used for the proper biquantized operator cf. (6.35); no confusion is to be expected.

The pseudo-biquantized operator $[\mathcal{P}]$ is not defined for arbitrary pseudo-antisymmetric pseudo-operators $\mathcal{P}$; the terms $P_{\cdot\cdot}$ of this pseudo-operator must satisfy appropriate conditions. The term $A^+P_{--}A^-$, the first contribution to $2[\mathcal{P}]$, is the biquantization of P_{--}; hence it is defined whenever P_{--} is. The last term $\mp A^-P_{++}A^+$, however, differs from the biquantization of P_{++} by the trace of $\mp P_{++}$,

$$\mp A^-P_{++}A^+ = A^+{}'P_{++}A^- \mp Tr\, P_{++}\,, \tag{23.23}$$

cf. Section 10. If P^s_{++} is an integral operator with the kernel $P_{++}(s', s'')$, the trace $Tr\,P_{++}$, provided it exists, can be expressed as

$$Tr\, P_{++} = \int P_{++}(s, s)\, dm(s). \tag{23.24}$$

The trace does exist if

$$\int |\, P_{++}(s, s)\,|\, dm(s) < \infty; \tag{23.25}$$

this condition evidently implies a severe restriction on the operator P_{++}. From the expression (23.24) we infer that the trace of the operator P_{++} equals that of its transposed $'P_{++}$,

$$Tr\, P_{++} = Tr\, 'P_{++}\,. \tag{23.26}$$

Since $'P_{++} = -P_{--}$ in virtue of the pseudo-antisymmetry of $\mathcal{P}$, cf. (23.14) the contribution (23.23) can also be written as

$$\mp A^-P_{++}A^+ = A^+P_{--}A^- \pm Tr\, P_{--}\,. \tag{23.27}$$

We take the liberty of calling an operator Λ "traceable" if its trace exists; we shall call it "square traceable" if the trace of the operator $\Lambda^*\Lambda$ exists. If Λ is square traceable its s-representer Λ^s is an integral operator with a kernel $\Lambda(s', s'')$ for which

$$Tr\, \Lambda^*\Lambda = \iint |\, \Lambda(s', s'')\,|^2\, dm(s')\, dm(s''). \tag{23.28}$$

Evidently,

$$Tr\, \Lambda\Lambda^* = Tr\, \Lambda^*\Lambda. \tag{23.29}$$

The contribution $A^-P_{+-}A^-$ to the field operator $2[\mathcal{P}]$ is defined if P^s_{+-} is an integral operator with a kernel $P_{+-}(s', s'')$ for which

$$\iint |\, P_{+-}(s', s'')\,|^2\, dm(s')\, dm(s'') < \infty. \tag{23.30$_1$}$$

This condition can now be expressed by saying that the operator P_{+-} should be square traceable. Clearly, the operator $A^{-}P_{+-}A^{-}$ transforms the two particle state Φ_2 with the representer $\psi_2(s_1, s_2)$ into the vacuum state with the representer $\psi_0 = \sqrt{2} \iint P_{+-}(s', s'') \psi_2(s', s'') \, dm(s') \, dm(s'')$, and this term is finite if condition $(23.30)_1$ is satisfied. One immediately verifies that under this condition the operator $A^{-}P_{+-}A^{-}$ is applicable to every n-particle state. To be sure, condition (23.26) is more severe than necessary. On the other hand, the operators P_{+-} (which will occur in the following) will be closely connected with the operators P_{-+}, and the condition

$$(23.30)_2 \qquad \iint |P_{-+}(s', s'')|^2 \, dm(s') \, dm(s'') < \infty$$

must be imposed on the kernel $P_{-+}(s', s'')$ of the operators P_{-+}; i.e. the operators P_{-+} must be square traceable. In fact, the operator $A^{+}P_{-+}A^{+}$ transforms the vacuum state with the representer $\psi_0 = 1$ into the two particle state with the representer

$$\psi_2(x_1, x_2) = \sqrt{2}\, P_{-+}(s_1, s_2)$$

and this state exists only if condition (23.29) is satisfied. However, it is easily verified that this condition is sufficient to insure that the operator $A^{+}P_{-+}A^{+}$ be applicable on every n-particle state.

We note that actually only the symmetric parts of P_{+-} and P_{-+} contribute to the operator $[\mathcal{P}]$ in case of a boson field, and only the antisymmetric parts in case of a fermion field; but for pseudo-antisymmetric pseudo-operators $\mathcal{P}$ the operators P_{+-} and P_{-+} are symmetric—or antisymmetric—see $(23.14)_2$.

Under the conditions imposed the field operator $[\mathcal{P}]$ can be applied to at least all states Φ which admit the number operator $N = A^{+}A^{-}$, see (10.8). For, as would not be difficult to show, a constant γ can be found such that the inequality

$$(23.31) \qquad \| [\mathcal{P}]\Phi \| \leq \gamma \, \| (N + 2)\Phi \|$$

holds for all such states Φ. By the process of closure the field operator $[\mathcal{P}]$ can be extended into a domain $\mathfrak{H}_{\mathcal{P}}$ in which it is closed.

If the pseudo-operator $\mathcal{R}$ is pseudo-hermitian, in addition to being pseudo-antisymmetric, *the field operator* $[\mathcal{R}]$ *is antihermitian*, i.e.,

$$(23.32) \qquad [\mathcal{R}]^* = -[\mathcal{R}].$$

For, the hermitian adjoints of $A^{+}R_{--}A^{-}$, $A^{+}R_{-+}A^{+}$, $\mp A^{-}R_{+-}A^{-}$, $\mp A^{-}R_{++}A^{+}$ are respectively $A^{+\prime}\overline{R}_{--}A^{-} = -A^{+}R_{--}A^{-}$, $A^{-\prime}\overline{R}_{-+}A^{-} = \pm A^{-}R_{+-}A^{-}$, $\mp A^{+\prime}\overline{R}_{+-}A^{+} = -A^{+}R_{-+}A^{+}$, $\mp A^{-\prime}\overline{R}_{++}A^{+} = \pm A^{-}R_{++}A^{+}$. Moreover, *the operator* $i[\mathcal{R}]$ *is hypermaximal* (or self-adjoint) *in the space* $\mathfrak{H}_{\mathcal{R}}$ as could be readily shown.

First Commutator Identity

The justification of the first form of the canonical transformation, which we shall now set up, will be derived from the evaluation of the commutator of the two field operators $w \circ \mathfrak{A}$ and $[\mathcal{P}] = \frac{1}{2} \mathfrak{A} \circ \mathcal{P}\mathfrak{A}$. This—proper—commutator is given by the identity

$$(23.33) \qquad [w \circ \mathfrak{A}, [\mathcal{P}]] = w \circ \mathcal{P}\mathfrak{A}$$

and is understood to hold when both sides are applied on a state which admits the operators $w \circ \mathfrak{A}\ [\mathcal{P}]$ and $[\mathcal{P}]\ w \circ \mathfrak{A}$; note that every state which admits the operator $N\sqrt{N}$ has this property. Symbolically, we may write identity (23.33) in the form

$$(23.34) \qquad [\mathfrak{A}, [\mathcal{P}]] = \mathcal{P}\mathfrak{A}.$$

In order to prove identity (23.33) we first employ identity (23.5) obtaining

$$(w \circ \mathfrak{A})(\mathfrak{A} \circ \mathcal{P}\mathfrak{A}) = \pm\mathfrak{A} \circ (w \circ \mathfrak{A})\mathcal{P}\mathfrak{A} + w \circ \mathcal{P}\mathfrak{A};$$

next we employ relations (23.3)′ and (23.5)′ obtaining

$$\begin{aligned} \pm\mathfrak{A} \circ (w \circ \mathfrak{A})\mathcal{P}\mathfrak{A} &= -\mathfrak{A} \circ (\mathfrak{A} \circ w)\mathcal{P}\mathfrak{A} \\ &= -'\mathcal{P}\mathfrak{A} \circ (\mathfrak{A} \circ w)\mathfrak{A} \\ &= \mp('\mathcal{P}\mathfrak{A} \circ \mathfrak{A})(\mathfrak{A} \circ w) - w \circ '\mathcal{P}\mathfrak{A} \\ &= (\mathfrak{A} \circ \mathcal{P}\mathfrak{A})(w \circ \mathfrak{A}) - w \circ '\mathcal{P}\mathfrak{A}. \end{aligned}$$

Addition of the two formulas yields

$$[w \circ \mathfrak{A}, (\mathfrak{A} \circ \mathcal{P}\mathfrak{A})] = w_0 \circ (\mathcal{P} - '\mathcal{P})\mathfrak{A}$$

whence (23.33) because of $'\mathcal{P} = -\mathcal{P}$.

Exponential Function of Pseudo-biquantized Operators

The desired canonical transformation will be given with the aid of the exponential function $\exp[\mathcal{P}]$ of pseudo-biquantized operators $[\mathcal{P}]$. If the operator $[\mathcal{P}]$ is pseudo-hermitian so that $i[\mathcal{P}]$ is hypermaximal, the function $\exp[\mathcal{P}]$ is defined as a unitary operator according to the theory of hypermaximal operators, cf. [19], [25]. In order to define the operator $\exp[\mathcal{P}]$ without assuming that $[\mathcal{P}]$ is pseudo-hermitian one may expand the exponential function in a power series and apply it first to states Φ' with only a finite number of components in the particle representation. The resulting sequence of components does not necessarily define a state $\Phi'' = \exp[\mathcal{P}]\Phi'$ with a finite norm. This will be clear from results obtained later on, cf. (24.47), (24.48), in connection with the case in which only the term P_{-+} is different from zero, so that $[\mathcal{P}]$ involves only creation operators. It can be shown, however, that the series $\exp[\mathcal{P}]$ is applicable on states Φ' provided the norms of the operators $P_{\bullet\bullet}$ are small enough. The norm $\|P\|$ of an operator P acting on states Ψ of a single particle is the smallest number p such that the inequality $\|P\Psi\| \leq p\|\Psi\|$ holds for all states Ψ. After the operator $\exp[\mathcal{P}]$ has been defined for the states Φ' it can be extended by the process of closure to a dense subset of the Hilbert space of all states Φ.

The subsequent operations involving operators of the form exp $[\mathcal{P}]$ could be justified—unless otherwise stated—provided the norms of the operators $P_{..}$ are small enough. It remains to justify these operations assuming only that the operators exp $[\mathcal{P}]$ involved are defined.

First Similarity Rule, First Form of the Operator T

For the field operator exp $[\mathcal{P}]$ formed with the aid of the pseudo-biquantized operator $[\mathcal{P}]$ we can form the field operator exp $[\mathcal{P}]$. For this operator we shall establish the identity

$$(23.35)\qquad \exp[-\mathcal{P}](w \circ \mathfrak{A}) \exp[\mathcal{P}] = w \circ (\exp \mathcal{P})\mathfrak{A},$$

valid for arbitrary functions $w(s)$. Since the left member of this formula represents a similarity transformation of the field operator $(w \circ \mathfrak{A})$ we call this formula the first "similarity rule." Symbolically, identity (23.35) may be written in the form

$$(23.36)\qquad \exp[-\mathcal{P}]\mathfrak{A} \exp[\mathcal{P}] = (\exp \mathcal{P})\mathfrak{A}.$$

If we specify $\mathcal{P}$ as the pseudo-operator $\mathfrak{R}$ for which $(\exp \mathfrak{R})\ \mathfrak{A} = \mathfrak{B}$ in accordance with (23.19), (23.17), we realize from (23.36) that the field operator

$$(23.37)\qquad T_1 = \exp[\mathfrak{R}] = \exp \tfrac{1}{2}\mathfrak{A} \circ \mathfrak{R}\mathfrak{A}$$

satisfies the relation

$$(23.38)\qquad \mathfrak{A}T_1 = T_1\mathfrak{B},$$

which is the same as (22.34). Since the field operator $[\mathfrak{R}]$ is anti-hermitian, cf. (23.32), the field operator T *is unitary* and can be extended to the whole Hilbert space $\mathfrak{H}$. Thus we see that *the field operator* $T_1 = \exp[\mathfrak{R}]$ *gives the desired canonical transformation.*

We consider the field operator

$$I(t) = \exp[t\mathcal{P}]w \circ (\exp t\mathcal{P})\mathfrak{A} \exp[-t\mathcal{P}],$$

and compute its derivative with respect to t formally; we obtain

$$\begin{aligned}\exp[-t\mathcal{P}]I'(t) \exp[t\mathcal{P}]\\ &= [\mathcal{P}]w \circ \exp(t\mathcal{P})\mathfrak{A} + w \circ (\exp t\mathcal{P})\mathcal{P}\mathfrak{A} - w \circ (\exp t\mathcal{P})\mathfrak{A}[\mathcal{P}]\\ &= \exp(t\mathcal{P})\{w' \circ \mathcal{P}\mathfrak{A} - [(w' \circ \mathfrak{A}), [\mathcal{P}]]\}\end{aligned}$$

with $w' = (\exp t\,'\mathcal{P})w$. According to identity (23.33), the operator in the curly brackets vanishes. Hence $I'(t) = 0$ and therefore $I(t) = I(0)$. Since, evidently, $I(0) = w \circ \mathfrak{A}$ the relation $I(t) = w \circ \mathfrak{A}$ is established.

24. *E-Ordering of the Canonical Transformation*

Specific information about the effects of the canonical transformation T cannot be derived directly from the expression exp $\tfrac{1}{2}\ \mathfrak{A} \circ \mathfrak{R}\mathfrak{A}$ for this transformation, see (23.37); for, creation and annihilation operators are still "interlocked" in this expression. The separation of these two types of operators will be performed in this section and various conclusions about the character of the transformation T will be drawn.

We shall use canonical transformations of the form

$$T = \exp[\mathcal{P}_1] \exp[\mathcal{P}_2] \cdots \exp[\mathcal{P}_n] \tag{24.1}$$

with appropriate pseudo-antisymmetric operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\cdots$. As an immediate consequence of the first similarity rule (23.36) we observe that for such a transformation the formula

$$T^{-1}\mathcal{A}T = \exp\mathcal{P}_1 \exp\mathcal{P}_2 \cdots \exp\mathcal{P}_n \mathcal{A} \tag{24.2}$$

holds. In order to ensure that the transformation T satisfies the desired condition

$$T^{-1}\mathcal{A}T = \mathcal{Y}\mathcal{A} = \mathcal{B} \tag{24.3}$$

it is, therefore, sufficient to ensure that the pseudo-operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\cdots$, $\mathcal{P}_n$, satisfy the relation

$$\exp\mathcal{P}_1 \exp\mathcal{P}_2 \cdots \exp\mathcal{P}_n = \mathcal{Y}. \tag{24.4}$$

Special efforts, on the other hand, are needed to show that the operator T is unitary.

Second Commutator Identity

In deriving the decomposition (24.1) we shall use the identity

$$[[\mathcal{P}_1], [\mathcal{P}_2]] = [[\mathcal{P}_1, \mathcal{P}_2]] \tag{24.5}$$

which expresses the fact that the *commutator of two pseudo-biquantized operators is the pseudo-biquantization of their commutator*. This identity may also be written in the form

$$[\mathcal{A} \circ \mathcal{P}_1\mathcal{A}, \mathcal{A} \circ \mathcal{P}_2\mathcal{A}] = 2\mathcal{A} \circ [\mathcal{P}_1, \mathcal{P}_2]\mathcal{A}. \tag{24.5$_1$}$$

Of course, the identity is meant to be valid when applied to a state which admits the operators $[\mathcal{P}_1][\mathcal{P}_2]$, $[\mathcal{P}_2][\mathcal{P}_1]$, and $[[\mathcal{P}_1, \mathcal{P}_2]]$.

Identity (24.5) can be verified by deriving three identities from the first commutator identity (23.33): first

$$\begin{aligned}(\mathcal{A} \circ \mathcal{P}_1\mathcal{A})[\mathcal{P}_2] &= ('\mathcal{P}_1\mathcal{A} \circ \mathcal{A})[\mathcal{P}_2] \\ &= ('\mathcal{P}_1\mathcal{A} \circ [\mathcal{P}_2]\mathcal{A}) + '\mathcal{P}_1\mathcal{A} \circ \mathcal{P}_2\mathcal{A},\end{aligned}$$

then

$$\begin{aligned}('\mathcal{P}_1\mathcal{A} \circ [\mathcal{P}_2]\mathcal{A}) &= (\mathcal{A} \circ [\mathcal{P}_2]\mathcal{P}_1\mathcal{A}) \\ &= [\mathcal{P}_2](\mathcal{A} \circ \mathcal{P}_1\mathcal{A}) + \mathcal{P}_2\mathcal{A} \circ \mathcal{P}_1\mathcal{A},\end{aligned}$$

and finally, by addition,

$$[(\mathcal{A} \circ \mathcal{P}_1\mathcal{A}), [\mathcal{P}_2]] = \mathcal{A} \circ (\mathcal{P}_1\mathcal{P}_2 + '\mathcal{P}_2\mathcal{P}_1)\mathcal{A};$$

thus (24.5) follows because of $'\mathcal{P}_2 = -\mathcal{P}_2$. (Incidentally, the relation $'\mathcal{P}_1 = -\mathcal{P}_1$ has not been used here.)

Composition Rule

Our process of decomposition will be derived from the following fact: *Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_{12}$ be three pseudo-antisymmetric pseudo-operators which satisfy the relation*

$$\exp \mathcal{P}_1 \exp \mathcal{P}_2 = \exp \mathcal{P}_{12} \; ; \tag{24.6}$$

then the corresponding pseudo-biquantized operators satisfy the corresponding relation

$$\exp [\mathcal{P}_1] \exp [\mathcal{P}_2] = \exp [\mathcal{P}_{12}]. \tag{24.7}$$

This formula will be referred to as the *composition rule for pseudo-biquantized operators*. We have not been able to obtain a complete proof for this rule; we are, therefore, forced to use it only heuristically. The second form of the canonical transformation T which we shall derive from it can, however, be verified independently.

It is possible to give a "formal" derivation of the composition rule, assuming that the exponential functions are formal power series. Multiplication of such formal series is to be performed term by term and rearrangement of the order of terms in a finite or infinite series is permitted but the order of multiplication can not be interchanged since the terms are not assumed to commute.

The term "Lie polynomial" of order $n > 1$ is used for every homogeneous polynomial in $\mathcal{P}_1$ and $\mathcal{P}_2$ which can be constructed from Lie polynomials of lower order solely by forming commutators and linear combinations with constant coefficients. A Lie polynomial of order 1 is any linear function of $\mathcal{P}_1$ and $\mathcal{P}_2$. Any formal series of Lie polynomials is called "Lie function" and is denoted by $l(\mathcal{P}_1, \mathcal{P}_2)$.

We can form Lie polynomials and Lie functions of pseudo-operators $\mathcal{P}_1$, $\mathcal{P}_2$ as well as of field operators $[\mathcal{P}_1]$, $[\mathcal{P}_2]$. An immediate consequence of the second commutator identity (24.5) is that a Lie function $l([\mathcal{P}_1], [\mathcal{P}_2])$ of two pseudo-biquantized operators is the pseudo-biquantization $[l(\mathcal{P}_1, \mathcal{P}_2)]$ of the Lie function $l(\mathcal{P}_1, \mathcal{P}_2)$

$$l([\mathcal{P}_1], [\mathcal{P}_2]) = [l(\mathcal{P}_1, \mathcal{P}_2)]. \tag{24.8}$$

We make use of an important fact discovered independently by Baker and Hausdorff [1], [2] in 1905: *The function* log (exp $\mathcal{P}_1$) (exp $\mathcal{P}_2$) *is a Lie function.*[8]

[8] Baker and Hausdorff [42,43] also gave a formula to describe this function explicitly in terms of commutators, but we do not need it here. For details see also a paper by W. Magnus [47].

The theorem of Baker and Hausdorff is an immediate consequence of the following characterization of Lie functions $f(\mathcal{P}_1, \mathcal{P}_2)$:

Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_1'$, $\mathcal{P}_2'$ be four elements such that both $\mathcal{P}_1$ and $\mathcal{P}_2$ commute with both $\mathcal{P}_1'$ and $\mathcal{P}_2'$; then

$$f(\mathcal{P}_1 + \mathcal{P}_1', \mathcal{P}_2 + \mathcal{P}_2') = f(\mathcal{P}_1, \mathcal{P}_2) + f(\mathcal{P}_1', \mathcal{P}_2').$$

For a proof of this fact see a forthcoming paper by Magnus in which various aspects and consequences of the theorem of Baker and Hausdorff are treated.

As a consequence of this theorem, formula (23.42) implies the formula

$$\log (\exp [\mathcal{P}_1])(\exp [\mathcal{P}_2]) = [\log (\exp \mathcal{P}_1)(\exp \mathcal{P}_2)].$$

Setting

$$\log (\exp \mathcal{P}_1)(\exp \mathcal{P}_2) = \mathcal{P}_{12}$$

in agreement with (24.6), we obtain

$$\log (\exp [\mathcal{P}_1])(\exp [\mathcal{P}_2]) = [\mathcal{P}_{12}]$$

and thus identity (24.7).

As a *Corollary to the composition rule* (24.7) we formulate: Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_{123}$ be four pseudo antisymmetric pseudo-operators which satisfy the relation

$$(24.6)' \qquad \exp \mathcal{P}_1 \exp \mathcal{P}_2 \exp \mathcal{P}_3 = \exp \mathcal{P}_{123} ;$$

then the corresponding pseudo-biquantized operators satisfy the relation

$$(24.7)' \qquad \exp [\mathcal{P}_1] \exp [\mathcal{P}_2] \exp [\mathcal{P}_3] = \exp [\mathcal{P}_{123}].$$

This corollary is an immediate consequence of identity (24.7) whenever a pseudo-operator $\mathcal{P}_{12}$ exists such that (24.6) holds. Such a pseudo-operator evidently exists as a formal series and the corollary again follows formally from the Baker-Hausdorff Theorem.

Second Form of the Canonical Transformation

The first form (23.37) of the canonical transformation $T_1 = \exp [\mathcal{R}] = \exp \frac{1}{2}\mathcal{A} \circ \mathcal{R}\mathcal{A}$ does not yet allow one to carry out the transformation of the unmodified into the modified representers in a direct way. For, since annihilation and creation operators are still "interlocked" in this form, the determination of a modified component $\chi_m(s)_m$ contributed by an unmodified component $\psi_n(s)_n$ still requires the evaluation of an infinite series.

It is possible to cast the transformation T into a different form in which annihilation and creation operators are separated.

It is necessary to introduce three special types of pseudo-antisymmetric pseudo-operators. If the three coefficients E_{--}, E_{++}, E_{-+} vanish,

$$(24.9)_1 \qquad E_{--} = E_{++} = E_{-+} = 0,$$

for such a pseudo-operator we denote it by $\mathcal{E}$. We then set

$$(24.9)_2 \qquad E_{+-} = E,$$

so that

$$(24.9)_3 \qquad \mathcal{E} = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}.$$

By (23.22) we have

(24.10) $$[\mathcal{E}] = \tfrac{1}{2}\mathfrak{A} \circ \mathcal{E}\mathfrak{A} = \mp\tfrac{1}{2}A^-EA^-.$$

Note that this expression involves only annihilation operators.

We denote by $\mathcal{G}$ a pseudo-antisymmetric pseudo-operator for which

(24.11)$_1$ $$G_{--} = G_{++} = G_{+-} = 0,$$

and set

(24.11)$_2$ $$G_{-+} = G,$$

so that

(24.11)$_3$ $$\mathcal{G} = \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix}.$$

Because of (23.22), the expression

(24.12) $$[\mathcal{G}] = \tfrac{1}{2}\mathfrak{A} \circ \mathcal{G}\mathfrak{A} = \tfrac{1}{2}A^+GA^+$$

involves only creation operators.

The condition that the pseudo-operators $\mathcal{E}$ and $\mathcal{G}$ be pseudo-antisymmetric is satisfied exactly if both E and G are symmetric for boson fields, antisymmetric for fermion fields, cf. (23.14)$_2$,

(24.13) $$'E = \pm E, \qquad 'G = \pm G.$$

We denote by $\mathcal{F}$ a pseudo-antisymmetric pseudo-operator for which

(24.14)$_1$ $$F_{+-} = F_{-+} = 0$$

and set

(24.14)$_2$ $$F_{--} = F.$$

Because of (23.14)$_1$ we have

(24.14)$_3$ $$F_{++} = -'F;$$

hence

(24.14)$_4$ $$\mathcal{F} = \begin{pmatrix} F & 0 \\ 0 & -'F \end{pmatrix}.$$

From (23.22) and (23.27) we derive

(24.15) $$[\mathcal{F}] = \tfrac{1}{2}\mathfrak{A} \circ \mathcal{F}\mathfrak{A} = A^+FA^- \pm \tfrac{1}{2}\, Tr\, F.$$

Thus, except for the additive constant $\pm\tfrac{1}{2}\, Tr\, F$, the pseudo-biquantized operator $[\mathcal{F}]$ is exactly the proper biquantized operator $[F]$.

The *second form of the canonical trasnformation* can now be defined as

$$T_2 = \exp [\mathcal{E}] \exp [\mathcal{F}] \exp [\mathcal{G}] \tag{24.16$_1$}$$

or

$$T_2 = \exp \{\mp \tfrac{1}{2}\, Tr\, F\} \exp \{\mp \tfrac{1}{2} A^- E A^-\} \exp \{A^+ F A^-\} \exp \{\tfrac{1}{2} A^+ G A^+\}. \tag{24.16$_2$}$$

The inverse operator T^{-1} is then of the form

$$\begin{aligned} T_2^{-1} &= \exp \{\pm \tfrac{1}{2}\, Tr\, F\} \\ &\cdot \exp \{-\tfrac{1}{2} A^+ G A^+\} \exp \{-A^+ F A^-\} \exp \{\pm \tfrac{1}{2} A^- E A^-\}. \end{aligned} \tag{24.16$_3$}$$

If an operator is given in this form it will be said to be *E-ordered*.[4] Here the quantification E is meant to refer to "exponential function."

In order to establish this form we must show that for every pseudo-antisymmetric pseudo-hermitian operator $\mathcal{R}$ there exist three pseudo-operators $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ of the type described such that the identity

$$\exp [\mathcal{E}] \exp [\mathcal{F}] \exp [\mathcal{G}] = \exp [\mathcal{R}] \tag{24.17}$$

holds.

Such a decomposition of the field operator $\exp [\mathcal{R}]$ can now be simply effected by performing the corresponding decomposition for the pseudo-operator $\exp \mathcal{R}$ itself,

$$\exp \mathcal{E} \exp \mathcal{F} \exp \mathcal{G} = \exp \mathcal{R}. \tag{24.18}$$

This fact is evidently implied by the Corollary to the Composition Rule formulated in the previous subsection, see (24.7)'. We shall show in a later subsection that the decomposition (24.18) can be effected very easily.

The advantage in using the second form of the transformation T becomes apparent if we desire to determine the N-representation of a state Φ whose M-representation is given. To this end we form the state Φ' whose N-representers are the same as the M-representers of Φ and set

$$\Phi = T^{-1}\Phi' = \exp [-\mathcal{G}] \exp [-\mathcal{F}] \exp [-\mathcal{E}]\Phi'. \tag{24.19}$$

Let us first assume that Φ is an eigenstate of the modified number M with the eigenvalue m such that the only non-vanishing M-representer is $\chi_m(s)_m$. This function is then the only non-vanishing N-component of Φ'. We expand the

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[4]This *E*-ordering is different from the *S*-ordering, sometimes also called well-ordering, of an operator; see [49]. An *S*-ordered operator is given as an infinite series of which each term is a homogeneous polynomial in creation and annihilation operators, the annihilation operators acting first, and the creation operators afterwards. The series expansions of $\exp \{-[\mathcal{E}]\}$ and $\exp \{[\mathcal{G}]\}$ lead to *S*-ordered operators but not the expansion of $\exp \{-[\mathcal{F}]\}$. Since the latter operator commutes with the number $N = A^+A^-$, it is immediately applicable as it stands, cf. (25.40), (25.41), while derivation and application of its *S*-ordered form requires a few involved operations.

exponential function $\exp[-\mathcal{E}]$ in a power series and apply each term on Φ'. Evidently, since $\mathcal{E}$ involves only annihilation operators, only a finite number, at most $m + 1$, terms of this series give a non-vanishing contribution. Since $\exp[-\mathcal{F}] = \exp\{-A^{+}FA^{-}\}\exp\{\mp \frac{1}{2} Tr\, F\}$, it is clear that the field operator $\exp[-\mathcal{F}]$ commutes with the number N and transforms the n-th component of $\exp[-\mathcal{E}]\,\Phi'$ into another n-particle component. The application of the field operator $\exp[-\mathcal{G}]$ produces components for every number n, but each such component consists of only a finite number—at most $m + 1$—terms. Thus the N-representers of the state Φ are easily calculated.

Modified Vacuum State

Let Φ be the modified vacuum state Φ^{B}_{vac} with the M-representers $\chi_0 = 1$, $\chi_m = 0$ for $m > 0$. Then the corresponding state Φ' is just the unmodified vacuum state Φ^{A}_{vac} and we may express Φ^{B}_{vac} in terms of Φ^{A}_{vac} through

$$\Phi^{B}_{vac} = T^{-1}\Phi^{A}_{vac},$$

cf. (23.27). Evidently,

$$\exp[-\mathcal{E}]\Phi^{A}_{vac} = \Phi^{A}_{vac}$$

since all terms but the first in the series $\sum_{\nu=0}^{\infty} \nu\,!\,[-\mathcal{E}]^{\nu}$ transform Φ^{A}_{vac} into zero. For the same reason in view of (24.15) we have

$$\exp[-\mathcal{F}]\Phi^{A}_{vac} = \eta\Phi^{A}_{vac}$$

if we set

(24.20) $$\eta = \exp\{\mp\tfrac{1}{2}\, Tr\, F\}.$$

Hence we are left with the relation

$$\Phi^{B}_{vac} = \eta \exp[-\mathcal{G}]\Phi^{A}_{vac}$$

or

(24.21) $$\Phi^{B}_{vac} = \eta \exp\{-\tfrac{1}{2}A^{+}GA^{+}\}\Phi^{A}_{vac}.$$

A simple calculation now shows that the n-th representer of the state Φ^{B}_{vac} is

(24.22) $$\psi^{B}_{n}(s)_n = \eta[n!]^{1/2}(n/2)!(-2)^{-n/2}\begin{pmatrix}\mathrm{Sy}\\ \mathrm{Asy}\end{pmatrix} g(s_n, s_{n-1}) \cdots g(s_2, s_1),$$

$$\psi^{B}_{0} = \eta,$$

provided n is even, and

(24.23) $$\psi^{B}_{n}(s)_n = 0$$

if n is odd.

In accordance with condition $(23.30)_2$ we must assume that the operator

G is square traceable so that G^S is an integral operator with a kernel $g(s', s'')$ for which

$$(24.24) \qquad Tr\, GG^* = \iint |g(s', s'')|^2\, dm(s')\, dm(s'') < \infty .$$

The fact that the norm of the state Φ^B_{vac} is unity, which follows from the unitary character of T, is expressed by the relation

$$(24.25) \qquad \begin{aligned} 1 + \sum_{\nu=1}^{\infty} (2\nu)![\nu!]^2(-2)^{-2\nu} \int \left| \begin{pmatrix} \mathrm{Sy} \\ \mathrm{Asy} \end{pmatrix} g(s_{2\nu}, s_{2\nu-1}) \cdots g(s_2, s_1) \right|^2 dm(s)_{2\nu} \\ = |\eta|^{-2} = \exp \{\pm \Re e\, Tr\, F\}. \end{aligned}$$

Below, see (24.38), we shall derive the relation

$$\Re e\, Tr\, F = -\tfrac{1}{2}\, Tr \log (1 \mp GG^*),$$

which will enable us to evaluate the infinite series (24.25).

The significance of the number $|\eta|$ is, of course, clear from the statement that

$$(24.26) \qquad |\eta|^2 = Pr\,(N, 0 \mid M, 0)$$

is *the probability that the field will be found in the unmodified vacuum state after it was found to be in the modified vacuum state.*

The probability that 2ν unmodified particles are found in the modified vacuum state is

$$(24.26)_1 \qquad \begin{aligned} Pr\,(N, 2\nu \mid M, 0) = \eta^2(2\nu)!(\nu!)^2(-2)^{-2\nu} \\ \cdot \int \left| \begin{pmatrix} \mathrm{Sy} \\ \mathrm{Asy} \end{pmatrix} g(s_{2\nu}, s_{2\nu-1}) \cdots g(s_2, s_1) \right|^2 dm(s)_{2\nu}, \end{aligned}$$

as seen from (24.23). This expression can also be evaluated as the term of order ν in the expansion of $\exp \{\mp \tfrac{1}{2}\, Tr \log (1 \mp GG^*)\}$ with respect to powers of GG^*.

The possibility of expressing the transformation of the unmodified into the modified vacuum state with the aid of an operator $\exp \{-\tfrac{1}{2} A^+GA^+\}$ as in (24.21) was suggested by G. Goertzel.[5] Once this transformation is found, the transformation of the M-representers into the N-representers of any state Φ can be determined. One expresses the state Φ in terms of the state Φ^B_{vac} with the aid of the modified creation operators B^+, expressing these in terms of A^-, and Φ^B_{vac} in terms of Φ^A_{vac}. In this way the state Φ is obtained from Φ^A_{vac} with the aid of operators built up in terms of A^-. The N-representers of Φ can then be found.

In fact, it would not be difficult to determine the operator G solely from the condition

$$B^- \exp \{-\tfrac{1}{2} A^+GA^+\}\Phi^A_{vac} = 0$$

and to proceed as indicated.

[5] In connection with Sollfrey's thesis [50]. This suggestion was the starting point for the investigations presented in this Part V.

Nevertheless, we prefer to present our procedure of E-ordering because it leads to easily managed explicit formulas.

First Decomposition

In order to effect the decomposition (24.18) of the pseduo-operator $\exp \mathcal{R}$, we first note that the pseudo-operators $\exp \mathcal{E}$ and $\exp \mathcal{G}$ reduce to

$$\exp \mathcal{E} = 1 + \mathcal{E}, \qquad \exp \mathcal{G} = 1 + \mathcal{G}; \tag{24.27}$$

for, as seen from (24.9) and (24.11), $\mathcal{E}^2 = 0$ and $\mathcal{G}^2 = 0$. Explicitly we have

$$\exp \mathcal{E} = \begin{pmatrix} 1 & 0 \\ E & 1 \end{pmatrix}, \qquad \exp \mathcal{G} = \begin{pmatrix} 1 & G \\ 0 & 1 \end{pmatrix}, \tag{24.27$_1$}$$

and

$$\exp \mathcal{F} = \begin{pmatrix} \exp F & 0 \\ 0 & \exp(-'F) \end{pmatrix}, \tag{24.28}$$

according to (24.14). A simple calculation then gives

$$\exp \mathcal{E} \exp \mathcal{F} \exp \mathcal{G} = \begin{pmatrix} \exp F & (\exp F)G \\ E \exp F & E(\exp F)G + \exp(-'F) \end{pmatrix}$$

and comparison with

$$\exp \mathcal{R} = \mathcal{Y} = \begin{pmatrix} Y_{--} & Y_{-+} \\ Y_{+-} & Y_{++} \end{pmatrix}$$

yields the relations

$$\begin{aligned} \exp F &= Y_{--}, \\ (\exp F)G &= Y_{-+}, \\ E \exp F &= Y_{+-}, \\ E \exp FG + \exp(-'F) &= Y_{++}. \end{aligned} \tag{24.29}$$

Thus the operator F is determined as

$$F = \log Y_{--}, \tag{24.30}$$

and G and E as

$$G = Y_{--}^{-1} Y_{-+}, \tag{24.31$_1$}$$

$$E = Y_{+-} Y_{--}^{-1}. \tag{24.31$_2$}$$

The last relation (24.29) is automatically satisfied; for, as will be seen, it is a consequence of the conditions $\mathcal{Y}'\mathcal{Y} = 1$ and $'\mathcal{Y}\mathcal{Y} = 1$. The latter conditions also insure that E and G satisfy the symmetry conditions (24.13). In fact, relation

$$\mp Y_{--}'Y_{-+} + Y_{-+}'Y_{--} = 0$$

implied by $\mathcal{Y}'\mathcal{Y} = 1$, cf. (24.9), yields $'G = \pm G$, and the relation

$$\mp 'Y_{+-}Y_{--} + 'Y_{--}Y_{+-} = 0$$

implied by $'\mathcal{Y}\mathcal{Y} = 1$ yields $'E = \pm E$. Setting $G = \pm 'G$, in the last relation (24.29) we may write it in the form

$$\pm Y_{+-}'Y_{-+}('Y_{--})^{-1} + ('Y_{--})^{-1} = Y_{++};$$

hence this relation follows from the relation

$$Y_{++}'Y_{--} \mp Y_{+-}'Y_{-+} = 1$$

implied by $\mathcal{Y}'\mathcal{Y} = 1$.

With these considerations we have derived the second, E-ordered, form (24.16) of the canonical transformation T^{-1} from the Composition Rule, provided the operator $\log Y_{--}$ can be formed. It is worth noting that the single particle operators E, F, G entering this second form are expressed directly in terms of the operators $Y_{\bullet\bullet}$ and not in terms of the pseudo-operator $\mathcal{R} = \log \mathcal{Y}$. Thus the second form of T can be described independently of the assumption that the pseudo-operator $\mathcal{Y}$ can be written in the form $\mathcal{Y} = \exp \mathcal{R}$.

Since we have not proved the Composition Rule rigorously, we should verify directly that the operator T_2 has the desired properties. It was already stated, cf. (24.3), that the relation

$$T_2^{-1}\mathcal{A}T_2 = \mathcal{B}$$

is a consequence of the first similarity rule. It is apparently not so simple to prove the unitary character of T_2; however we shall do so in Section 25.

Relations between E, F, G, and $\mathcal{Y}$

Since various expressions will be needed later for the operators E and G in terms of the coefficients of $\mathcal{Y}$, they will be enumerated here:

$$G = Y_{--}^{-1}Y_{-+} = \pm 'Y_{-+}'Y_{--}^{-1} \tag{24.32}$$

$$\bar{G} = Y_{++}^{-1}Y_{+-} = \pm 'Y_{+-}'Y_{++}^{-1} \tag{24.33}$$

$$E = Y_{+-}Y_{--}^{-1} = \pm 'Y_{--}^{-1}'Y_{+-} \tag{24.34}$$

$$\bar{E} = Y_{-+}Y_{++}^{-1} = \pm 'Y_{++}^{-1}'Y_{-+}. \tag{24.35}$$

The first column is implied by (22.31); the second column follows from the symmetry property of E and G established above. Furthermore, we have

$$1 - G\bar{G} = ('Y_{++}Y_{--})^{-1} = Y_{--}^{-1}{}'Y_{++}^{-1}, \tag{24.36}$$

$$1 - \bar{G}G = ('Y_{--}Y_{++})^{-1} = Y_{++}^{-1}{}'Y_{--}^{-1}, \tag{24.37}$$

$$1 - E\bar{E} = (Y_{++}{}'Y_{--})^{-1} = {}'Y_{--}^{-1}Y_{++}^{-1}, \tag{24.38}$$

$$1 - \bar{E}E = (Y_{--}{}'Y_{++})^{-1} = {}'Y_{++}^{-1}Y_{--}^{-1}. \tag{24.39}$$

The first relation follows from

$$Y_{--}[1 - G\bar{G}]'Y_{++} = Y_{--}{}'Y_{++} \mp Y_{-+}{}'Y_{+-} = 1$$

by (24.32), (24.33) and $\mathcal{Y}'\underline{\mathcal{Y}} = \underline{1}$. The other relations are derived similarly.

Since the operators $\pm\bar{G} = '\bar{G} = G^*$ and $\pm\bar{E} = '\bar{E} = E^*$ are the Hermitian adjoints of the operators G and E, it is clear that the operators $1 - G\bar{G}$, $1 - \bar{G}G$, $1 - E\bar{E}$, $1 - \bar{E}E$ are non-negative and hypermaximal. Therefore these operators possess non-negative square roots. For example, we may introduce a non-negative operator ${}_1F$ through the relation

$$\exp {}_1F = ['Y_{++}Y_{--}]^{1/2} = [1 - G\bar{G}]^{-1/2}. \tag{24.40$_1$}$$

We may also introduce an anti-hermitian operator ${}_0F$ by means of

$$\exp {}_0F = Y_{--}['Y_{++}Y_{--}]^{1/2}. \tag{24.40$_2$}$$

Evidently,

$$\exp {}_0F^* = ['Y_{++}Y_{--}]^{-1/2}{}'Y_{++} = ['Y_{++}Y_{--}]^{1/2}Y_{--}^{-1} = \exp\{-{}_0F\}.$$

Consequently, the operator $i\,{}_0F$ is hermitian, in fact it is hypermaximal. Because of

$$\exp F = Y_{--}, \tag{24.29$_1$}$$

relation

$$\exp {}_0F \exp {}_1F = \exp F \tag{24.41}$$

holds.

Trace Relations

We maintain that relation (24.41) implies relation

$$\exp Tr\,{}_0F \exp Tr\,{}_1F = \exp Tr\,F. \tag{24.42}$$

This *trace composition rule*[6] is a—rather simple—analogue of the general compo-

[6] If the operator F acts in a finite dimensional space the expression $\exp Tr \log F$ reduces to the determinant of the matrix F^{8}. The trace composition rule (24.42) is, therefore, the analogue of the rule that the determinant of the product of two matrices is the product of their determinants.

sition rule (24.7). Like the general composition rule, it follows from the Baker-Hausdorff Theorem; for, the trace of a commutator of two operators is the commutator of the traces of these operators, namely zero.

Since the operator $i\, {}_0F$ is hermitian its trace is real; i.e., $Tr\, {}_0F$ is imaginary. Consequently, $|\exp Tr\, {}_0F| = 1$ so that

$$|\exp Tr\, F| = \exp Tr\, {}_1F \tag{24.43}$$

or, by $(24.40)_1$,

$$\mathcal{Re}\, Tr\, F = Tr\, {}_1F = -\tfrac{1}{2}\, Tr \log [1 - G\bar{G}]. \tag{24.44}$$

The absolute value of the number η introduced earlier, see (24.20), can now be expressed as

$$|\eta| = \exp \{\mp\tfrac{1}{2}\, Tr\, {}_1F\}$$

or, by $(24.40)_1$ and (24.44), as

$$\begin{aligned} |\eta| &= \exp \{\mp\tfrac{1}{2}\, Tr \log {}'Y_{++}Y_{--}\} \\ &= \exp \{\pm\tfrac{1}{2}\, Tr \log [1 - G\bar{G}]\}. \end{aligned} \tag{24.45}$$

Since, obviously, the operator $\mp\log [1 - G\bar{G}] = \mp\log [1 \mp GG^*]$ is positive definite, its trace is positive, and consequently $|\eta| < 1$, in agreement with (24.25). Incidentally, formula (24.45) gives an evaluation of the infinite series (24.25),

$$\begin{aligned} 1 + \sum_{\nu=1}^{\infty} (2\nu!)[\nu!]^2(-2)^{2\nu} & \\ \cdot \int \left| \begin{pmatrix}\mathrm{Sy}\\ \mathrm{Asy}\end{pmatrix} g(s_{2\nu}, s_{2\nu-1}) \cdots g(s_2, s_1) \right|^2 dm(s)_{2\nu} & \\ = \exp \frac{1}{2} \sum_{\nu=1}^{\infty} (\pm 1)^{\nu-1}\nu^{-1} & \\ \cdot \int g(s_1, s_{2\nu})\overline{g(s_{2\nu}, s_{2\nu-1})} \cdots g(s_3, s_2)\overline{g(s_2, s_1)}\, dm(s)_{2\nu}, & \end{aligned} \tag{24.46}$$

which could perhaps be derived directly.[7]

Conditions for the Existence of the Canonical Transformation

In conclusion, we summarize the conditions under which the second form of the canonical transformation $T_2 = \exp [\mathcal{E}] \exp [\mathcal{F}] \exp [\mathcal{G}]$ exists.

We have assumed that the operators Y_{--} are bounded and that the pseudo-operator $\mathcal{Y}$ is pseudo-hermitian and satisfies the conditions $\mathcal{Y}'\mathcal{Y} = {}'\mathcal{Y}\mathcal{Y} = 1$;

[7]This formula is related to the evaluation of an infinite series derived by Feynman [4] with the aid of his graphs.

moreover, we have tacitly assumed that the operators Y_{--} and Y_{++} possess inverses. As a consequence, relation (24.36) holds and, since Y^*_{--} and Y_{--} are bounded, the operator $1 \mp G\,G^*$ has a positive lower bound. This fact may be expressed by the relation

$$(24.47) \qquad \| GG^* \| < 1$$

in the boson case, if we denote by $\| GG^* \|$ the least upper bound of $(\psi, GG^*\psi) = \| G^* \psi \|^2$ for all functions ψ in $\mathfrak{H}$ with $\| \psi \| = 1$; similarly, $\| E^*E \| < 1$ holds in the boson case. For the existence of the operators $[\mathcal{G}]$ and $[\mathcal{E}]$ additional restrictions must be imposed, namely that the operators G and E possess square traces:

$$(24.48) \qquad Tr\, GG^* < \infty \quad , \quad Tr\, E^*E < \infty .$$

We maintain that under this condition combined with (24.47) the operator $\mp\log\,[1 \mp GG^*]$ also possesses a finite trace. This can be deduced most easily from the fact that the condition $Tr\, GG^* < \infty$ implies that the operator GG^* has a pure point spectrum of eigenvalues $\gamma^2 \geq 0$ for which $\sum \gamma^2 = Tr\, GG^*$. As a consequence it is seen that

$$Tr\, \log\,[1 + GG^*] = \sum \log\,[1 + \gamma^2] \leq \sum \gamma^2;$$

hence we may conclude that

$$(24.49)_- \qquad Tr\, \log\,[1 + GG^*] \leq Tr\, GG^*$$

in the fermion case. Furthermore, we have

$$-Tr\, \log\,[1 - GG^*] = -\sum \log\,[1 - \gamma^2] \leq [1 - \gamma^2_{max}]^{-1} \sum \gamma^2,$$

and since $\gamma^2_{max} = \| GG^* \|$, we may conclude that

$$(24.49)_+ \qquad -Tr\, \log\,[1 - GG^*] \leq [1 - \| GG^* \|]^{-1}\, Tr\, GG^*$$

in the boson case.

The conditions (24.47), (24.48) are somewhat stronger than necessary for the existence of the transformation T_3 ; however, the conditions $Tr\, GG^* < \infty$ and $Tr\, E^*E < \infty$ are necessary.

If these conditions are not satisfied, the field does not possess a particle representation with respect to the operators $B^\mp$ as creation and annihilation operators. Nevertheless, it would seem likely that the field possesses an occupation number representation of the second type in the sense explained in Section 19, Part IV. The total weight associated with this representation would then be

$$(24.50) \qquad W = \mp\tfrac{1}{2}\, Tr\, \log\,[1 \mp GG^*]$$

and hence infinite if $Tr\, GG^* = \infty$; in other words, if $T_rGG^* = \infty$ the field would be "myriotic" with respect to the operators $B^\mp$. Although it would be of interest to follow up this possibility, it will not be done here.

25. *Third and Fourth Form of the Canonical Transformation*

In this section we shall establish two further forms of the transformation T. The third form involves only the operators G, ${}_0F$ and their conjugates; the fourth form results from the third by E-ordering. It is possible to show that the fourth form is identical with the third as well as with the second without using the composition rule. The third form is evidently unitary and thus it follows that the others are also unitary.

The main reason for discussing these other forms is not so much that we want to prove the identity of the first and second forms of the operator T. It is rather that we want to have the opportunity to formulate and apply other similarity and decomposition rules. The content of the present Section 25 is not essentially used in the subsequent sections.

Third and Fourth Form of the Transformation T

The third form of the canonical transformation will be derived from a "second" decomposition

$$\mathcal{Y} = \exp {}_0\mathfrak{F} \exp {}_1\mathcal{R} \tag{25.1}$$

of the pseudo-operator $\mathcal{Y} = \exp \mathcal{R}$ in which the pseudo-operator

$$ {}_0\mathfrak{F} = \begin{pmatrix} {}_0F & 0 \\ 0 & {}_0\bar{F} \end{pmatrix} \tag{25.2}$$

possesses only diagonal terms while terms of this type are absent from the pseudo-operator

$$ {}_1\mathcal{R} = \begin{pmatrix} 0 & {}_1R_{-+} \\ {}_1R_{+-} & 0 \end{pmatrix}. \tag{25.2$_1$}$$

These pseudo-operators ${}_0\mathfrak{F}$ and ${}_1\mathcal{R}$, which we shall describe below, will be seen to be pseudo-antisymmetric and pseudo-hermitian, cf. (23.14) and (23.16). Having found the decomposition (25.1) we shall obtain from the composition identity (24.7) the *third form*

$$T_3 = \exp [{}_0\mathfrak{F}] \exp [{}_1\mathcal{R}] \tag{25.3}$$

of the canonical transformation T in which, by (23.22), (23.27),

$$\exp [{}_0\mathfrak{F}] = \exp \{\mp \tfrac{1}{2}\, Tr\, {}_0F\} \exp \{A^+\, {}_0F A^-\} \tag{25.4}$$

and

$$\exp [{}_1\mathcal{R}] = \exp \{\tfrac{1}{2}A^+\, {}_1R_{-+}\, A^+ \mp \tfrac{1}{2}A^-\, {}_1R_{+-}\, A^-\}. \tag{25.4$_1$}$$

Because of the pseudo-hermitian character of ${}_0\mathfrak{F}$ and ${}_1\mathcal{R}$, both operators $\exp [{}_0\mathfrak{F}]$

and $\exp\,[{}_1\mathcal{R}]$ are unitary; hence it is evident that the operator $\exp\,[{}_0\mathcal{F}]\,\exp\,[{}_1\mathcal{R}]$ is unitary.

The fourth form of the operator T will be obtained by decomposing the pseudo-operator $\exp\,{}_1\mathcal{R}$ as

$$\exp\,{}_1\mathcal{R} = \exp\,{}_1\mathcal{E}\,\exp\,{}_1\mathcal{F}\,\exp\,{}_1\mathcal{G} \tag{25.5}$$

corresponding to the first decomposition (24.18). The composition identity, cf. (24.17), then gives the *fourth form*

$$T_4 = \exp\,[{}_0\mathcal{F}]\,\exp\,[{}_1\mathcal{E}]\,\exp\,[{}_1\mathcal{F}]\,\exp\,[{}_1\mathcal{G}] \tag{25.6}$$

of the canonical transformation, where, as in the second form (24.16), annihilation and creation operators are separated.

We shall show that the identity of the fourth form of T with both the third and second forms can be verified without using the composition identity—which was not proved, but only derived formally. Since the third form is evidently unitary the same is true of the E-ordered second and fourth forms once their identity is established. Note that the pseudo-operators $\mathcal{E}$, $\mathcal{G}$ and ${}_1\mathcal{E}$, ${}_1\mathcal{G}$ are—in general—not pseudo-hermitation and that therefore—in general—the operators $\exp\,[\mathcal{E}]$, $\exp\,[\mathcal{G}]$ and $\exp\,[{}_1\mathcal{E}]$, $\exp\,[{}_1\mathcal{G}]$ are not unitary.

Various other advantages of the fourth form will be discussed later on.

Second Decomposition

In order to effect the second decomposition (25.1) of the pseudo-operator

$$\mathcal{Y} = \begin{pmatrix} Y_{--} & Y_{-+} \\ Y_{+-} & Y_{++} \end{pmatrix},$$

see (23.6), we first decompose it in the form

$$\mathcal{Y} = {}_0\mathcal{Y}\,{}_1\mathcal{Y} \tag{25.7}$$

with

$$\tag{25.8$_0$} {}_0\mathcal{Y} = \begin{pmatrix} 'Y_{++}^{-1}['Y_{++}Y_{--}]^{1/2} & 0 \\ 0 & 'Y_{--}^{-1}['Y_{--}Y_{++}]^{1/2} \end{pmatrix}$$

and

$$\tag{25.8$_1$} {}_1\mathcal{Y} = \begin{pmatrix} ['Y_{++}Y_{--}]^{1/2} & ['Y_{++}Y_{--}]^{1/2}Y_{--}^{-1}Y_{-+} \\ ['Y_{--}Y_{++}]^{1/2}Y_{++}^{-1}Y_{+-} & ['Y_{--}Y_{++}]^{1/2} \end{pmatrix}.$$

The two diagonal terms in the pseudo-operator ${}_0\mathcal{Y}$ are evidently unitary since $\overline{Y}_{--} = Y_{++}$, $'\overline{Y}_{++} = {}'Y_{--}$. The first of these terms is in fact identical with the right member of relation $(24.40)_2$. The two terms can therefore be written in the form $\exp\,{}_0F$ and $\exp\,{}_0\overline{F}$ with ${}_0F = \log\,\{Y_{--}['Y_{++}Y_{--}]^{1/2}\}$. Consequently, the pseudo-operator ${}_0\mathcal{Y}$ is of the form

$$(25.9) \qquad {}_0\mathcal{Y} = \exp {}_0\mathcal{T},$$

where the pseudo-operator ${}_0\mathcal{T}$ is given by (25.2) and $(24.40)_2$.

Evidently, the pseudo-operator ${}_0\mathcal{T}$ is pseudo-hermitian and pseudo-anti-symmetric and hence ${}_0\mathcal{Y}$ is unitary.

The elements of the pseudo-operator ${}_1\mathcal{Y}$ can easily be expressed in terms of the operators

$$(25.10)_- \qquad G = Y_{--}^{-1}Y_{-+} = \pm'Y_{-+}'Y_{--}^{-1}$$

and

$$(25.10)_+ \qquad \bar{G} = Y_{++}^{-1}Y_{+-} = \pm'Y_{+-}'Y_{++}^{-1},$$

cf. (23.32), (23.33). In fact, using the formulas

$$(25.11) \qquad 'Y_{++}Y_{--} = [1 - G\bar{G}]^{-1/2}, \qquad 'Y_{--}Y_{++} = [1 - \bar{G}G]^{-1/2},$$

cf. (24.36), (24.37), and noting that $G^* = \pm\bar{G}$ we may write

$$(25.12) \qquad {}_1\mathcal{Y} = \begin{pmatrix} [1 - G\bar{G}]^{-1/2} & [1 - G\bar{G}]^{-1/2}G \\ [1 - \bar{G}G]^{-1/2}\bar{G} & [1 - \bar{G}G]^{-1/2} \end{pmatrix}.$$

We now maintain that his pseudo-operator ${}_1\mathcal{Y}$ is of the form

$$(25.13) \qquad {}_1\mathcal{Y} = \exp {}_1\mathcal{R}$$

if we choose for ${}_1\mathcal{R}$ the pseudo-operator

$$(25.14) \qquad {}_1\mathcal{R} = \begin{pmatrix} 0 & h(G\bar{G})G \\ h(\bar{G}G)\bar{G} & 0 \end{pmatrix},$$

with

$$h(z) = z^{-1/2} \operatorname{arctanh} z^{1/2}.$$

This statement may be proved by a power series expansion of $\cosh {}_1\mathcal{R}$ and $\sinh {}_1\mathcal{R}$, $\operatorname{arctanh} z^{1/2}$ making use of the identities

$$(25.15)_1 \qquad \bar{G}f(G\bar{G}) = f(\bar{G}G)\bar{G},$$

$$(25.15)_2 \qquad f(G\bar{G})G = Gf(\bar{G}G),$$

which are evident for polynomials f and then follow for functions f for which the operators $f(G\bar{G})$ and $f(\bar{G}G)$ are defined. Thus the second decomposition of $\mathcal{Y}$ is effected.

The pseudo-operators ${}_1\mathcal{E}$, ${}_1\mathcal{T}$, ${}_1\mathcal{G}$ of the final decomposition (25.5) can be obtained from the terms of the pseudo-operator ${}_1\mathcal{Y}$ as given by (25.12) with the aid of the formulas (24.30), (24.31) by substituting the terms of ${}_1\mathcal{Y}$ instead of those of $\mathcal{Y}$. We find

$$ {}_1F = -\tfrac{1}{2} \log [1 - G\bar{G}] \tag{25.16} $$

$$ {}_1G = G, \qquad {}_1E = \bar{G}; \tag{25.17} $$

in deriving (25.17), identity $(25.15)_1$ is to be used. The pseudo-operators ${}_1\mathcal{G}$, ${}_1\mathcal{E}$, ${}_1\mathcal{F}$ are then

$$ {}_1\mathcal{G} = \begin{pmatrix} 0 & G \\ 0 & 1 \end{pmatrix}, \qquad {}_1\mathcal{E} = \begin{pmatrix} 0 & 0 \\ \bar{G} & 0 \end{pmatrix}, \tag{25.18} $$

$$ {}_1\mathcal{F} = -\frac{1}{2}\begin{pmatrix} \log [1 - G\bar{G}] & 0 \\ 0 & -\log [1 - \bar{G}G] \end{pmatrix} \tag{25.19} $$

and their exponential functions are

$$ \exp {}_1\mathcal{G} = \begin{pmatrix} 1 & G \\ 0 & 1 \end{pmatrix}, \qquad \exp {}_1\mathcal{E} = \begin{pmatrix} 1 & 0 \\ \bar{G} & 1 \end{pmatrix}, \tag{25.20} $$

$$ \exp {}_1\mathcal{F} = \begin{pmatrix} [1 - G\bar{G}]^{-1/2} & 0 \\ 0 & [1 - \bar{G}G]^{1/2} \end{pmatrix}. \tag{25.21} $$

Final Form of the Transformation T

The fourth form of the transformation T can now finally be written as

$$ \begin{aligned} T_4 = {} & \exp \{\mp\tfrac{1}{2}\, Tr\, {}_0F\} \exp \{\pm\tfrac{1}{2}\, Tr \log [1 - \bar{G}G]\} \exp \{A^+\, {}_0FA^-\} \\ & \cdot \exp \{\mp\tfrac{1}{2}A^-\bar{G}A^-\} \exp \{A^+ \log [1 - G\bar{G}]^{1/2}A^-\} \exp \{\tfrac{1}{2}A^+GA^+\}. \end{aligned} \tag{25.22} $$

Since the operator ${}_0F$ is anti-hermitian the first factor here is a number of absolute value 1 and therefore, rather insignificant. The second factor, on the other hand, is a positive number less than one. Actually, this factor is simply the absolute value of the number η introduced above, see (24.45).

Identification of the Third and Fourth Form of T

In order to prove the identity of the third and fourth forms of the canonical transformation, we replace G and $\bar{G}$ by tG and $t\bar{G}$ in ${}_1\mathcal{E}$, ${}_1\mathcal{G}$, ${}_1\mathcal{F}$ as given by (25.18), (25.19), letting t be a real parameter. We note that the derivatives of ${}_1\mathcal{E}$, ${}_1\mathcal{G}$, ${}_1\mathcal{F}$ with respect to t are given by

$$ {}_1\dot{\mathcal{E}} = t^{-1}\, {}_1\mathcal{E}, \qquad {}_1\dot{\mathcal{G}} = t^{-1}\, {}_1\mathcal{G}, \tag{25.23} $$

$$ {}_1\dot{\mathcal{F}} = \begin{pmatrix} [1 - t^2G\bar{G}]tG\bar{G} & 0 \\ 0 & -[1 - t^2\bar{G}G]t\bar{G}G \end{pmatrix}. \tag{25.24} $$

Evidently, each of the pseudo-operators ${}_1\mathcal{E}$, ${}_1\mathcal{G}$, ${}_1\mathcal{F}$ commutes with its derivative. The derivatives of the exponential functions are therefore given by

$$(25.25)\qquad \frac{d}{dt}\exp {}_1\mathcal{E} = {}_1\dot{\mathcal{E}} \exp {}_1\mathcal{E}, \qquad \frac{d}{dt}\exp {}_1\mathcal{G} = {}_1\dot{\mathcal{G}} \exp {}_1\mathcal{G},$$

$$(25.26)\qquad \frac{d}{dt}\exp {}_1\mathcal{F} = {}_1\dot{\mathcal{F}} \exp {}_1\mathcal{F}.$$

We observe that the derivative

$$(25.27)\qquad {}_1\dot{\mathcal{R}} = \begin{pmatrix} 0 & [1 - t^2G\bar{G}]^{-1}G \\ [1 - t^2\bar{G}G]^{-1}\bar{G} & 0 \end{pmatrix}$$

of the pseudo-operator ${}_1\mathcal{R}$, given by (25.14), also commutes with ${}_1\mathcal{R}$ so that

$$\frac{d}{dt}\exp {}_1\mathcal{R} = {}_1\dot{\mathcal{R}} \exp {}_1\mathcal{R}.$$

Since, by (25.5)

$$\exp {}_1\mathcal{E} \exp {}_1\mathcal{F} \exp {}_1\mathcal{G} \exp(-{}_1\mathcal{R}) = 1,$$

the derivative of this pseudo-operator with respect to t vanishes. Hence we obtain the relation

$$(25.28)\qquad \begin{aligned} &\exp(-{}_1\mathcal{G}) \exp(-{}_1\mathcal{F})_1\dot{\mathcal{E}} \exp {}_1\mathcal{F} \exp {}_1\mathcal{G} \\ &\qquad + \exp(-{}_1\mathcal{G})_1\dot{\mathcal{F}} \exp {}_1\mathcal{G} + {}_1\dot{\mathcal{G}} - {}_1\dot{\mathcal{R}} = 0, \end{aligned}$$

which, of course, could also be verified directly.

Now we consider the operator

$$K = \exp[{}_1\mathcal{E}] \exp[{}_1\mathcal{F}] \exp[{}_1\mathcal{G}] \exp[-{}_1\mathcal{R}];$$

our aim is to prove that it is the identity. Again we replace G and $\bar{G}$ by tG and $t\bar{G}$ and differentiate with respect to t. As before, the drivative of each of the exponents $[{}_1\mathcal{E}]$, $\cdots$, $[{}_1\mathcal{R}]$ commutes with the exponent. Consequently, the derivative $\dot{K}$ of K is given by

$$\dot{K} = \exp[{}_1\mathcal{E}] \exp[{}_1\mathcal{F}] \exp[{}_1\mathcal{G}] Z \exp[-{}_1\mathcal{R}]$$

with

$$\begin{aligned} Z = &\exp[-{}_1\mathcal{G}] \exp[-{}_1\mathcal{F}][{}_1\dot{\mathcal{E}}] \exp[{}_1\mathcal{F}] \exp[{}_1\mathcal{G}] \\ &\qquad + \exp[-{}_1\mathcal{G}][{}_1\dot{\mathcal{F}}] \exp[{}_1\mathcal{G}] + [{}_1\dot{\mathcal{G}}] - [{}_1\dot{\mathcal{R}}]. \end{aligned}$$

Now we make use of the "second similarity rule" which we shall prove in the next subsection. It states that the relation

$$\exp(-\mathcal{P}_1)\mathcal{P}_2 \exp \mathcal{P}_1 = \mathcal{P}_2$$

between pseudo-antisymmetric pseudo-operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ entails the relation

$$\exp[-\mathcal{P}_1][\mathcal{P}_2]\exp[\mathcal{P}_1] = [\mathcal{P}_3]$$

for the corresponding pseudo-biquantized operators. From this rule in conjunction with relation (25.28) we immediately infer

$$Z = 0$$

and hence $\dot{K} = 0$. Consequently, K is constant, and since K reduces to the identity for $t = 0$, the desired statement $K = 1$ follows. Thus the identity of the third and the fourth forms of the operator T is established.

Second and Third Similarity Rule

The *second similarity rule* which states that the relation

$$\exp(-\mathcal{P}_1)\mathcal{P}_2\exp\mathcal{P}_1 = \mathcal{P}_3 \tag{25.29}$$

between three pseudo-antisymmetric operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ entails the relation

$$\exp[-\mathcal{P}_1][\mathcal{P}_2]\exp[\mathcal{P}_1] = [\mathcal{P}_3] \tag{25.30}$$

between the corresponding biquantized operators can be proved in the same way as the first similarity rule (23.35). One replaces $\mathcal{P}_1$ by $t\mathcal{P}_1$ and differentiates the operator $J = \exp[t\mathcal{P}_1][\exp(-t\mathcal{P}_1)\mathcal{P}_2\exp[t\mathcal{P}_1]\exp[-t\mathcal{P}_1]$ with respect to t, obtaining

$$\exp[t\mathcal{P}_1]\{[[\mathcal{P}_1], [\exp(-t\mathcal{P}_1)\mathcal{P}_2\exp t\mathcal{P}_1]]$$
$$- [\exp(-t\mathcal{P}_1)[\mathcal{P}_1, \mathcal{P}_2]\exp t\mathcal{P}_1]\}\exp[-t\mathcal{P}_1].$$

This expression vanishes since, by the second commutator rule (24.5),

$$[[\mathcal{P}_1], [\exp(-t\mathcal{P}_1)\mathcal{P}_2\exp t\mathcal{P}_1]] = [[\mathcal{P}_1, \exp(-t\mathcal{P}_1)\mathcal{P}_2\exp t\mathcal{P}_1]]$$
$$= [\exp(-t\mathcal{P}_1)[\mathcal{P}_1, \mathcal{P}_2]\exp t\mathcal{P}_1].$$

Consequently, the operator J is independent of t and hence equal to $[\mathcal{P}_2]$ since it reduces to this operator for $t = 0$. In order to make this reasoning complete, one should, of course, supply arguments of the type needed to establish the first similarity rule.

From the second similarity rule one immediately infers the fact that relation (25.29) implies the relation

$$\exp[-\mathcal{P}_1]f([\mathcal{P}_2])\exp[\mathcal{P}_1] = f[\mathcal{P}_3] \tag{25.31}$$

for every polynomial f. The same relation also holds for functions f for which $f([\mathcal{P}_2])$ and $f([\mathcal{P}_3])$ can be defined. In particular, we are led to the *third similarity rule*: Relation (25.29) implies relation

$$\exp[-\mathcal{P}_1]\exp[\mathcal{P}_2]\exp[\mathcal{P}_1] = \exp[\mathcal{P}_3]. \tag{25.32}$$

It is interesting to observe that relation (25.29) implies relation

$$\exp(-\mathcal{P}_1)f(\mathcal{P}_2)\exp\mathcal{P}_1 = f(\mathcal{P}_3); \tag{25.33}$$

this relation is immediately verified for polynomials f and therefore holds also for more general functions f. In particular, we have

$$\exp(-\mathcal{P}_1) \exp \mathcal{P}_2 \exp \mathcal{P}_1 = \exp \mathcal{P}_3 . \tag{25.34}$$

In conjunction with this relation, identity (25.32) becomes a special case of the corollary to the composition rule (24.7)'. This special case is thus derived independently of the—unproved—general composition rule.

We shall use the third similarity rule in establishing the identity of the fourth and the second forms of the transformation T.

Composition Rule for Biquantized Operators

We shall need another special case of the composition rule, which also can be proved without the general rule, namely the case in which the pseudo-antisymmetric pseudo-operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_{12}$ have only diagonal terms; i.e., they are of the form

$$\mathfrak{F} = \begin{pmatrix} F & 0 \\ 0 & -'F \end{pmatrix}. \tag{25.35}$$

The rule then states that relation

$$\exp \mathfrak{F}_1 \exp \mathfrak{F}_2 = \exp \mathfrak{F}_{12} \tag{25.36}$$

for three such pseudo-operators entails relation

$$\exp [\mathfrak{F}_1] \exp [\mathfrak{F}_2] = \exp [\mathfrak{F}_{12}]. \tag{25.37}$$

Clearly, relation (25.36) consists of the pair of relations

$$\exp F_1 \exp F_2 = \exp F_{12} \tag{25.38}$$

and

$$\exp(-'F_1) \exp(-'F_2) = \exp(-'F_{12}); \tag{25.38$'$}$$

the second is evidently a consequence of the first. By (23.22), (23.27) the composition formula (25.37) can be written as

$$\exp \{A^+F_1A^- \mp \tfrac{1}{2} Tr\, F_1\} \exp \{A^+F_2A^- \mp \tfrac{1}{2} Tr\, F_2\}$$
$$= \exp \{A^+F_{12}A^- \mp \tfrac{1}{2} Tr\, F_{12}\};$$

because of the composition rule for traces (24.41), (24.42), it is equivalent with the composition rule

$$\exp \{A^+F_1A^-\} \exp \{A^+F_2A^-\} = \exp \{A^+F_{12}A^-\} \tag{25.39}$$

for ordinary biquantized operators. It is, therefore, sufficient to derive formula (25.39) from (25.38).

For this purpose we shall use the particle representation of the states Φ,

$$\Phi \leftrightarrow \{\psi_n(s)_n\}.$$

The representation

$$A^+FA^-\Phi \leftrightarrow \{(F^{\cdot 1} + \cdots + F^{\cdot n})\psi_n(s)_n\}$$

of the state $A^+FA^-\Phi$, cf. (6.35), involves the operators $F^{\cdot\nu}$ which act on the function $\psi_n(s)_n$ only in so far as this function depends on s_ν. In the representation of exp $\{A^+FA^-\}$ Φ,

$$\text{(25.40)} \qquad \exp\{A^+FA^-\}\Phi \leftrightarrow \{\exp(F^{\cdot 1} + \cdots + F^{\cdot n})\psi_n(s)_n\},$$

cf. (6.36), we may write

$$\text{(25.41)} \qquad \exp(F^{\cdot 1} + \cdots + F^{\cdot n}) = \exp(F^{\cdot 1}) \cdots \exp(F^{\cdot n}).$$

since the operators $F^{\cdot 1}, \cdots, F^{\cdot n}$ commute.

Moreover, the operators $F_1^{\cdot\alpha}$ and $F_2^{\cdot\beta}$, formed from two operators F_1 and F_2, commute even if F_1 and F_2 do not, provided $\alpha \neq \beta$; for, the two operators then act on functions of different variables. Consequently, we have the identity

$$\exp(F_1^{\cdot 1} + \cdots + F_1^{\cdot n}) \exp(F_2^{\cdot 1} + \cdots + F_2^{\cdot n})$$
$$= \exp F_1^{\cdot 1} \cdots \exp F_1^{\cdot n} \exp F_2^{\cdot 1} \cdots \exp F_2^{\cdot n}$$
$$= \exp F_1^{\cdot 1} \exp F_2^{\cdot 1} \cdots \exp F_1^{\cdot n} \exp F_2^{\cdot n}.$$

Applying relation (25.38) and relation (25.41) to $F = F_{12}$ we obtain

$$\text{(25.42)} \quad \exp(F_1^{\cdot 1} + \cdots + F_1^{\cdot n}) \exp(F_2^{\cdot 1} + \cdots + F_2^{\cdot n}) = \exp(F_{12}^{\cdot 1} + \cdots + F_{12}^{\cdot n}).$$

Since, according to (25.40), the states exp $\{A^+F_1A^-\}\Phi$, exp $\{A^+F_2A^-\}\Phi$ and exp $\{A^+F_{12}A^-\}\Phi$ have the representations

$$\exp\{A^+F_1A^-\} \exp\{A^+F_2A^-\}\Phi$$
$$\leftrightarrow \{\exp(F_1^{\cdot 1} + \cdots + F_1^{\cdot n}) \exp(F_2^{\cdot 1} + \cdots + F_2^{\cdot n})\psi_n(s)_n\}$$

and

$$\exp\{A^+F_{12}A^-\}\Phi \leftrightarrow \{\exp(F_{12}^{\cdot 1} + \cdots + F_{12}^{\cdot n})\psi_n(s)_n\},$$

identity (25.42) insures that these two states are the same. Thus the composition rule (25.39) is proved and rule (25.37) follows immediately.

It is remarkable that the derivation of the composition rule for biquantized operators from a representation makes its validity rather obvious while it does not seem obvious how to derive this rule by essentially algebraic means legalizing the use of Baker and Hausdorff's theorem.

Identity of the Fourth and Second Forms of the Operator T

In order to show that the fourth form

$$T_4 = \exp[{}_0\mathfrak{F}] \exp[{}_1\mathcal{E}] \exp[{}_1\mathfrak{F}] \exp[{}_1\mathcal{G}]$$

of the operator T, cf. (25.6), is identical with the second form

$$T_2 = \exp[\mathcal{E}] \exp[\mathfrak{F}] \exp[\mathcal{G}],$$

cf. (24.16), we first verify the identity

$$\exp {}_0\mathfrak{F} \exp {}_1\mathfrak{E} = \exp \mathfrak{E} \exp {}_0\mathfrak{F}. \tag{25.43}$$

This is immediately done by writing the pseudo-operator $\exp {}_0\mathfrak{F}$ in the form

$$\exp {}_0\mathfrak{F} = \begin{pmatrix} Y_{--}[1 - G\bar{G}]^{1/2} & 0 \\ 0 & Y_{++}[1 - \bar{G}G]^{1/2} \end{pmatrix}, \tag{25.44}$$

cf. (25.9), $(25.8)_0$, (25.11) and the pseudo-operators $\exp \mathfrak{E}$, $\exp {}_1\mathfrak{E}$ in the forms

$$\exp \mathfrak{E} = \begin{pmatrix} 1 & 0 \\ Y_{+-}Y_{--}^{-1} & 1 \end{pmatrix},$$

cf. (24.27), $(24.31)_2$,

$$\exp {}_1\mathfrak{E} = \begin{pmatrix} 1 & 0 \\ \bar{G} & 1 \end{pmatrix},$$

cf. (25.20), using identity $(25.15)_1$ and setting $\bar{G} = Y_{++}^{-1} Y_{+-}$ by (24.33).

The third similarity rule (25.32) is then applicable to $\mathcal{P}_1 = -{}_0\mathfrak{F}$, $\mathcal{P}_2 = {}_1\mathfrak{E}$ and $\mathcal{P}_3 = \mathfrak{E}$; combined with (25.6) it leads to the form

$$T_4 = \exp [\mathfrak{E}] \exp [{}_0\mathfrak{F}] \exp [{}_1\mathfrak{F}] \exp [{}_1\mathfrak{G}].$$

We next employ the decomposition formula

$$\exp {}_0\mathfrak{F} \exp {}_1\mathfrak{F} = \exp \mathfrak{F}, \tag{25.45}$$

which follows from (25.44), (25.21) and

$$\exp \mathfrak{F} = \begin{pmatrix} Y_{--} & 0 \\ 0 & 'Y_{--}^{-1} \end{pmatrix},$$

cf. (24.28), (24.30), (25.11). Now we can apply the composition rule (25.37) for biquantized operators and obtain the expression

$$T_4 = \exp [\mathfrak{E}] \exp [\mathfrak{F}] \exp [{}_1\mathfrak{G}]$$

for the fourth form of T, which is the same as the second form because ${}_1\mathfrak{G} = \mathfrak{G}$, see (25.17). Thus we have shown that the fourth form of T is identical with the second without using the general composition rule. Since the fourth form is identical with the third form which is evidently unitary, we conclude that the second form of T is unitary.

26. *Application to Boson Fields*

Having found the desired canonical transformation in general we proceed to discuss its specific nature in the original problem of a boson field with spring forces.

Reduction of the Quantized to the Unquantized Field Problem

The pseudo-operator $\mathcal{Y}$ associated with the boson field with spring forces is given by

(26.1)
$$Y_{--} = Y_{++} = \tfrac{1}{2}[\Upsilon^{1/2}\Omega^{-1/2} + \Upsilon^{-1/2}\Omega^{1/2}]$$
$$Y_{+-} = Y_{-+} = \tfrac{1}{2}[\Upsilon^{1/2}\Omega^{-1/2} - \Upsilon^{-1/2}\Omega^{1/2}],$$

cf. (22.22). Here Υ and Ω are the modified and unmodified energy operators for single particles, connected by the relation

(26.2)
$$\Upsilon^2 = \Omega^2 + Q$$

in which the operator Q corresponds to the "spring constant;" cf. (22.15).

It is therefore possible to express the operator $G = Y_{--}^{-1} Y_{-+}$ and various other operators that played a role in earlier discussions in terms of Ω and Υ. For this purpose it is convenient to introduce the operators

(26.3)
$$\Gamma = \Omega^{-1/2}\Upsilon\Omega^{-1/2}$$

and

(26.4)
$$Z = \Upsilon^{1/2}\Omega^{-1/2}\Gamma^{-1/2}.$$

Clearly, the operator Γ is Hermitian while Z is unitary; for,

$$Z^* = \Gamma^{-1/2}\Omega^{-1/2}\Upsilon^{1/2},$$

hence

$$Z^*Z = \Gamma^{-1/2}\Gamma\Gamma^{-1/2} = 1$$

and

$$ZZ^* = \Upsilon^{1/2}\Omega^{-1/2}(\Omega^{1/2}\Upsilon^{-1}\Omega^{1/2})\Omega^{-1/2}\Upsilon^{1/2} = 1.$$

The operators Y_{--} and Y_{-+} may be written in terms of Γ and Z as

(26.5)
$$Y_{--} = \tfrac{1}{2}Z\Gamma^{-1/2}[\Gamma + 1], \qquad Y_{-+} = \tfrac{1}{2}Z\Gamma^{-1/2}[\Gamma - 1],$$

the operator G, defined by (24.31), may be written simply as

(26.6)
$$G = [\Gamma + 1]^{-1}[\Gamma - 1]$$

and, since $\overline{G} = G$, the operator $1 - G\overline{G}$ becomes

(26.7)
$$1 - G\overline{G} = 4\Gamma[\Gamma + 1]^{-2}.$$

Finally we note that the unitary operator $\exp {}_0F$, defined by (24.40), is exactly the operator Z

(26.8)
$$Z = \exp {}_0F.$$

Suppose the operators Ω and Υ are "known," so that the operators Z, Γ, and hence ${}_0F$, $G = \overline{G}$, and so on, can be formed. Assuming that these operators satisfy the conditions formulated earlier, the canonical transformation T in any of its various forms can be set up. By expressing T in terms of Ω and Υ, we may also say that *the problem of modifying the quantized field has been reduced to the problem of modifying the unquantized field.*

Conditions for the Existence of the Canonical Transformation

Necessary and sufficient conditions to be imposed on the operators G and Z were discussed at the end of Section 24. These conditions are equivalent with restrictions on the disturbance operator Q; it does not seem to be easy, however, to give to these restrictions on Q a simple, directly verifiable, form. Nevertheless, certain general statements can be made about them; in doing this it is convenient to work with the operator Γ instead of G.

The conditions $\|G\bar{G}\| < 1$ and $Tr\, G\bar{G} < \infty$, which were introduced at the end of Section 24, cf. (24.47), (24.48), are essentially equivalent with the condition

$$Tr\,(\Gamma - 1)^2 < \infty, \tag{26.9}$$

as could be derived from the formula

$$\Gamma - 1 = 2G[1 - G]^{-1}. \tag{26.10}$$

The operators Υ and Q are easily expressed in terms of the operators Γ and Ω by

$$\Upsilon = \Omega^{1/2}\Gamma\Omega^{1/2} = \Omega^{1/2}(\Gamma - 1)\Omega^{1/2} + \Omega \tag{26.11}$$

and

$$\begin{aligned} Q &= \Omega^{1/2}\Gamma\Omega\Gamma\Omega^{1/2} - \Omega^2 \\ &= \Omega^{1/2}(\Gamma - 1)\Omega(\Gamma - 1)\Omega^{1/2} + \Omega^{3/2}(\Gamma - 1)\Omega^{1/2} + \Omega^{1/2}(\Gamma - 1)\Omega^{3/2}. \end{aligned} \tag{26.12}$$

Special Cases

The requirements on the operator Γ formulated above exclude certain types of disturbance or—in the case of several media—interaction. For example, *homogeneous interaction is excluded.*

We say an operator Λ is "homogeneous" if it commutes with the momentum; for example, Λ is homogeneous if its x-representer is an integral operator with a kernel $\Lambda(x', x'')$ which depends only on the difference $x' - x''$, i.e. if

$$\Lambda(x', x'') = \Lambda(x' - x''). \tag{26.13}$$

Here it is implied that the field extends over the infinite space.

If the interaction operator Q is homogenous then Γ is also homogeneous; for, Q commutes with Ω in this case and the operator Γ is therefore given by $\Gamma = 1 + \Omega^{-2}Q$ as seen from (26.12). Evidently, the kernel $\gamma(x', x'') = \gamma(x' - x'')$ of the operator Γ does not satisfy the condition $Tr\,(\Gamma - 1)^2 < \infty$ since $\int\int \gamma^2\,(x' - x'')\,dx'\,dx'' = \infty$.

Another noteworthy fact is that *one-dimensional media with vanishing rest mass μ are excluded if the spring constants are non-negative everywhere.*

To verify this statement we may introduce the Fourier transforms $\tilde{q}(k', k'')$ and $\tilde{\gamma}(k', k'')$ of the kernels $q(x', x'')$ and $\gamma(x', x'')$ of the operators Q and $\Gamma - 1$. Since (26.12) holds and $\mu = 0$ these kernels are related by

$$\tilde{q}(k', k'') = |k'k''|^{1/2}(|k'| + |k''|)\tilde{\gamma}_2(k', k'') + |k'k''|^{1/2} \int \tilde{\gamma}(k', k) \, |k| \, \tilde{\gamma}(k, k'') \, dk.$$

Since the x- and k-spaces are assumed one-dimensional, condition

$$\iint |\tilde{\gamma}(k', k'')|^2 \, dk' \, dk'' < \infty$$

implies that $|k'k''|^{1/2} \tilde{\gamma}(k', k'')$ vanishes as $|k'|$ and $|k''| \to 0$. Consequently, $\tilde{q}(0, 0) = 0$. This relation now is equivalent with

$$\iint q(x', x'') \, dx' \, dx'' = 0$$

in contradiction to the assumption that $q(x', x'')$ is non-negative.

Since the failure is due to the behaviour of the function $\tilde{q}(k', k'')$ for $k' = k'' = 0$ we may say that a non-negative spring constant causes an infrared catastrophe in a one dimensional medium. One of the infinities occurring in Sollfrey's problem is due, essentially, to this fact.

Finally we remark that two- or more dimensional media are excluded if the springs which provide the interaction are attached to isolated points. Ultraviolet catastrophes would occur in such cases.

We now consider an example which is very unrealistic physically but can be handled explicitly. In this example the x-representer of the operator $\Upsilon - \Omega$ if of the form

(26.14) $$\zeta(x', x'') = \iota\zeta(x')\zeta(x''),$$

in which the real valued function $\zeta(x)$ and the constant ι are at our disposal. The case in which the interaction operator Q is itself of such a form could also be treated explicitly but would be more complicated and equally unrealistic.

Instead of the x-representers we prefer to work with the energy representers, setting

(26.15) $$\hat{\zeta}(\omega', \omega'') = \iota\hat{\zeta}(\omega')\hat{\zeta}(\omega'')$$

instead of (26.14). We assume that the energy is connected with the position x in a real manner. The energy range is $\mu < \omega < \infty$ where μ is the rest mass, which may be zero or positive. The condition that the operator with the kernel (26.15) produces quadratically integrable functions is

(26.16) $$\int_\mu^\infty |\hat{\zeta}(\omega)|^2 \, d\omega < \infty;$$

this condition must therefore be imposed. We must also require that the operator T be positive definite. This is automatically the case if $\iota > 0$, but if $\iota < 0$ we need the relation

$$-\iota \int_{\mu}^{\infty} \omega^{-1} \mid \hat{\zeta}(\omega) \mid^{2} d\omega < 1, \qquad \iota < 0. \tag{26.17}$$

From relation (26.11) we see that the kernel of the operator Γ^{Ω} is

$$\Gamma(\omega', \omega'') = \iota(\omega'\omega'')^{-1/2}\hat{\zeta}(\omega')\hat{\zeta}(\omega''),$$

provided this operator exists; this is the case if

$$\int_{\mu}^{\infty} \omega^{-1} \mid \hat{\zeta}(\omega) \mid^{2} d\omega < \infty. \tag{26.18}$$

Since the trace of the square of the operator Γ is exactly the square of the left member of (26.17), this operator is square traceable if it exists. By virtue of (26.17) this condition is automatically satisfied if $\iota < 0$ or if $\mu > 0$. If $\iota > 0$ and $\mu = 0$, however, functions $\hat{\zeta}(\omega)$ can be defined which satisfy condition (26.16) but not condition (26.18). Thus we see *that it may happen that the operator* T *is defined and has a spectrum of the same type as* Ω *and still the canonical transformation* T *does not exist.*

If $\iota < 0$ and $\mu > 0$ the operator T may possess a point eigenvalue υ_0. The condition for the occurrence of such a point eigenvalue is evidently the existence of a quadratically integrable function $\psi_0(\omega)$ for which

$$(\omega - \upsilon_0)\psi_0(\omega) + \iota\hat{\zeta}(\omega) \int_{\mu}^{\infty} \hat{\zeta}(\omega)\psi_0(\omega)\, d\omega = 0.$$

This condition is satisfied for

$$\psi_0(\omega) = c(\omega - \upsilon_0)^{-1}\hat{\zeta}(\omega) \tag{26.19}$$

provided the value $\upsilon = \upsilon_0$ can be so chosen in the interval $0 < \upsilon < \mu$ that

$$-\iota \int_{\mu}^{\infty} (\omega - \upsilon)^{-1} \mid \hat{\zeta}(\omega) \mid^{2} d\omega = 1. \tag{26.20}$$

When υ varies from 0 to μ the left member of this equation varies from the left member of condition (26.17) to the left member of condition

$$-\iota \int_{\mu}^{\infty} (\omega - \mu)^{-1} \mid \hat{\zeta}(\omega) \mid^{2} d\omega > 1, \qquad \iota < 0. \tag{26.18}$$

Clearly, if this latter condition is satisfied a point eigenvalue $\upsilon = \upsilon_0$ exists.

Thus we see that *it may happen that the spectra of the operators* T *and* Ω *are of different types and still the canonical transformation* T *does exist.*

The existence of a point eigenstate or "bound" state Ψ_0 of the operator T entails the existence of bound states of the modified field. If $\psi_0(x)$ is the representer of Ψ_0 and υ_0 the eigenvalue, the state Φ_n of the field with the particle representer $\psi_n(s)_n = \psi_0(s_1) \cdots \psi_0(s_n)$ is such a bound state with the eigenvalue $n\,\nu_0$. One may say that in this state the field consists of n particles in the state Ψ_0. Thus we see that *the canonical transformation may exist even in cases in which the modified field possesses bound states while the unmodified field does not.*

27. *Transition Operator. Scattering Operator*

In this section we shall calculate the transition amplitudes connecting the state of the field at two different times. We can do this in two ways: Either we can employ the "method of spectral transformation" making use of the fact that variation in time of the modified particle representers can immediately be determined by applying the operator $\exp\{-itH_s\}$. Or we may utilize the fact that the annihilation operators at two different times are connected by a transformation similar to that which connects modified and unmodified operators of this kind.

We shall also study the asymptotic behavior of the transition amplitudes and, in particular, give a simple asymptotic description of the incomplete scattering operator in terms of "original but scattered," "spontaneously emitted and half scattered," and "spontaneously emitted but not scattered" particles. We shall see that no complete scattering operator in the strict sense exists, but that nevertheless the transition probabilities approach definite limits as the time interval increases indefinitely.

Method of Spectral Transformation

We employ the canonical transformation T in order to express the particle representers $\psi_n(s)_n(t)$ at any time t in terms of the "$N(0)$-representers" $\psi_n(s)_n(0)$, see (22.27). We consider the functions $\psi_n(s)_n(t)$ as the $N(0)$-representers of a time-dependent Schrödinger state $\Phi_s(t)$, cf. Section 15 footnote 4 and (15.15). As seen from (22.28), (22.32), (22.17), this state is given by

$$\Phi_s(t) = T^{-1}\exp\{-itA^+\mathrm{T}A^-\}T\Phi. \tag{27.1}$$

Since the relation

$$\exp\{-itA^+\mathrm{T}A^-\}T = T\exp\{-itB^+\mathrm{T}B^-\} \tag{27.2}$$

can be derived from the relation $A^*T = TB^*$, see (22.34), the state $\Phi_s(t)$ can also be expressed as

$$\Phi_s(t) = \exp\{-itB^+\mathrm{T}B^-\}\Phi = \exp\{-itH_s\}\Phi, \tag{27.3}$$

and thus be recognized as the solution of the Schrödinger equation

$$i\nabla_t\Phi_s(t) = H_s\Phi_s(t). \tag{27.4}$$

Instead of using $\Phi_s(t)$ we prefer to use the corresponding interaction state $\Phi_w(t)$ given by

$$\Phi_w(t) = W(t)\Phi \tag{27.5}$$

when we set

$$(27.6) \qquad W(t) = \exp\{itA^{+}\Omega A^{-}\}T^{-1}\exp\{-itA^{+}\Upsilon A^{-}\}T.$$

or

$$(27.7) \qquad W(t) = \exp\{itA^{+}\Omega A^{-}\}\exp\{-itB^{+}\Upsilon B^{-}\}.$$

Transition Operator

Operators of the type $\exp A^{+}\Lambda A^{-}$ will occur occasionally in the following. Since we want to use our similarity and composition rules for pseudo-biquantized operators we introduce to every operator Λ the pseudo-antisymmetric pseudo-operator

$$(27.8) \qquad \hat{\Lambda} = \begin{pmatrix} '\Lambda & 0 \\ 0 & -'\Lambda \end{pmatrix}$$

and use the relation

$$(27.9) \qquad \exp[\hat{\Lambda}] = \exp\{\tfrac{1}{2}A\hat{\Lambda}A\} = \exp\{\tfrac{1}{2}\,Tr\,\Lambda\}\exp A^{+}\Lambda A^{-}.$$

In particular, although the traces $Tr\,\Omega$ and $Tr\,\Upsilon$ do not exist, we shall introduce the pseudo-operators $\hat{\Omega}$ and $\hat{\Upsilon}$ and furthermore the field operator

$$(27.10) \qquad \tilde{T}^{-1}(t) = \exp\{it[\hat{\Omega}]\}T^{-1}\exp\{-it[\hat{\Upsilon}]\}T,$$

which we shall call the "transition operator." We have reserved the notation $\tilde{T}(t)$ for its inverse, the operator

$$(27.10)_1 \qquad \begin{aligned} \tilde{T}(t) &= T^{-1}\exp\{it[\hat{\Upsilon}]\}T\exp\{-it[\hat{\Omega}]\} \\ &= \exp\{it[\hat{\Omega}]\}\tilde{T}^{-1}(-t)\exp\{-it[\hat{\Omega}]\} \end{aligned}$$

in order to facilitate the description of the decomposition formulas.

If the operator $\tilde{T}^{-1}(t)$ existed it would differ from the operator $W(t)$ by the factor $\exp\{-ith\}$, as seen from (27.9) for $\Lambda = \Omega$ and $\Lambda = \Upsilon$; i.e.,

$$(27.11) \qquad \tilde{T}^{-1}(t) = \exp\{-ith\}W(t),$$

with $h = \tfrac{1}{2}\,Tr\,(\Upsilon - \Omega)$, cf. (22.26). Assuming that this difference trace exists, we may define $\tilde{T}(t)$ by $(27.10)_1$ and employ our rules as we would if the expression $(27.10)_1$ were defined. The resulting identities must, however, be verified by some other method. The state

$$(27.12) \qquad \tilde{\Phi}_{*}(t) = \exp\{-ith\}\Phi_{*}(t) = \tilde{T}^{-1}(t)\Phi$$

will be called the "adjusted interaction state."

Another form of the transition operator is

$$(27.13) \qquad \tilde{T}^{-1}(t) = \exp\left\{\frac{it}{2}\,\mathcal{A}\circ\hat{\Omega}\mathcal{A}\right\}\exp\left\{-\frac{it}{2}\,\mathcal{B}\circ\hat{\Upsilon}\mathcal{B}\right\}.$$

this formula is identical with (27.10), as we may deduce from the relation

(27.14) $$[{}'\mathcal{Y}\hat{\mathrm{T}}\mathcal{Y}] = \tfrac{1}{2}\mathcal{B} \circ \hat{\mathrm{T}}\mathcal{B}$$

or

$$\mathcal{A} \circ {}'\mathcal{Y}\hat{\mathrm{T}}\mathcal{Y}\mathcal{A} = \mathcal{Y}\mathcal{A} \circ \hat{\mathrm{T}}\mathcal{Y}\mathcal{A} = \mathcal{B} \circ \hat{\mathrm{T}}\mathcal{B},$$

which follows from (22.21). We need only use the first form $T = T_1 = \exp\{[\log \mathcal{Y}]\}$ of T and the third similarity rule (25.32), (25.29), with $\mathcal{P}_1 = \log \mathcal{Y}$, $\mathcal{P}_2 = \hat{\mathrm{T}}$.

Direct Method

Expressions for the operator $\bar{T}(t)$ can also be derived by recognizing that the annihilation and creation operators $A^{\mp}(s, t)$ at the time t and those at the time $t = 0$ are connected by a homogeneous linear relation[8] of the same type as those which connect the operators $B^{\mp}$ with the operators $A^{\mp}$.

We note that relation (22.19), with s instead of x, can be written in the form

$$\mathcal{B}(s, t) = \exp\{-it\hat{\mathrm{T}}'\}\mathcal{B}(s, 0)$$

or simply as

(27.15) $$\mathcal{B}(t) = \exp\{-it\hat{\mathrm{T}}\}\mathcal{B}(0).$$

If we insert this expression into (22.24) and then insert the result into (22.21) for $t = 0$ we obtain, by (23.20)′,

(27.16) $$\mathcal{A}(t) = {}'\mathcal{Y}\exp\{-it\hat{\mathrm{T}}\}\mathcal{Y}\mathcal{A}(0).$$

Since we would like to employ our rules, we shall work with the "adjusted" annihilation and creation operators $\bar{A}^{\mp}(s, t)$, given by

(27.17) $$\bar{\mathcal{A}}(t) = \exp\{it\hat{\Omega}\}\mathcal{A}(t)$$

or

(27.18) $$\bar{\mathcal{A}}(t) = {}'\bar{\mathcal{Y}}(t)\mathcal{A}(0)$$

with

(27.19) $${}'\bar{\mathcal{Y}}(t) = \exp\{it\hat{\Omega}\}{}'\mathcal{Y}\exp\{-it\hat{\mathrm{T}}\}\mathcal{Y},$$

or

(27.19)$_1$ $$\begin{aligned}\bar{\mathcal{Y}}(t) &= {}'\mathcal{Y}\exp\{it\hat{\mathrm{T}}\}\mathcal{Y}\exp\{-it\hat{\Omega}\}\\ &= \exp\{it\hat{\Omega}\}{}'\bar{\mathcal{Y}}(-t)\exp\{-it\hat{\Omega}\}.\end{aligned}$$

[8] This is true even if the disturbing operator Q is not independent of the time. We shall not treat this case here, although it would be interesting to do so, in particular, since the results could presumably be cast into a Lorentz invariant form. This case will be treated in a forthcoming paper by B. Zumino.

The theory developed in the preceding sections can be applied to this transformation $\tilde{\mathcal{Y}}(t)$ since it evidently satisfies the condition $\tilde{\mathcal{Y}}(t)'\tilde{\mathcal{Y}}(t) = 1$. Without assuming that the operator $\mathcal{Y}(t)$ is of the form $\exp \tilde{\mathfrak{R}}(t)$, we may find the corresponding canonical transformation by the composition rule. On the right member of $(27.19)_1$ we replace $'\mathcal{Y}$, $\exp\{it\hat{\mathrm{T}}\}$, $\mathcal{Y}$, and $\exp\{-it\hat{\Omega}\}$ by $\exp[\log '\mathcal{Y}] = T^{-1}$, $\exp\{it[\hat{\mathrm{T}}]\}$, $\exp[\log \mathcal{Y}] = T$, and $\exp\{-it[\hat{\Omega}]\}$ respectively. The resulting canonical transformation is seen to be exactly the field operator $\tilde{T}(t)$, given by $27.10)_1$. Relation (27.18) leads therefore to relation

$$\tilde{\mathfrak{a}}(t) = \tilde{T}(t)\mathfrak{a}(0)\tilde{T}^{-1}(t) \tag{27.20}$$

from which we deduce that *the* $\mathfrak{a}(0)$*-representation of the adjusted interaction state* $\Phi_\bullet$, cf. (27.12), *gives a* $\tilde{\mathfrak{a}}(t)$*-representation of the state* Φ.

Properties of the Transition Operator

We proceed to calculate various quantities which will enable us to analyze the nature of the inverse transition operator $\tilde{T}(t)$. We shall in particular refer to the second form

$$\tilde{T}(t) = \exp[\tilde{\mathcal{E}}(t)] \exp[\tilde{\mathfrak{F}}(t)] \exp[\tilde{\mathfrak{G}}(t)] \tag{27.21}$$

of this operator, see (24.18). From (27.19) we find

$$\begin{aligned} \tilde{Y}_{--}(t) &= ['Y_{++} \exp\{it\mathrm{T}\}Y_{--} - 'Y_{-+} \exp\{-it\mathrm{T}\}Y_{+-}] \exp\{-it\Omega\} \\ \tilde{Y}_{-+}(t) &= ['Y_{++} \exp\{it\mathrm{T}\}Y_{-+} - 'Y_{-+} \exp\{-it\mathrm{T}\}Y_{++}] \exp\{it\Omega\} \end{aligned} \tag{27.22}$$

or, from (24.35), (24.34),

$$\tilde{Y}_{--}(t) = 'Y_{++}[\exp\{it\mathrm{T}\} - \bar{E}\exp\{-it\mathrm{T}\}E]Y_{--}\exp\{-it\Omega\},$$

$$\tilde{Y}_{-+}(t) = 'Y_{++}[\exp\{it\mathrm{T}\}\bar{E} - \bar{E}\exp\{-it\mathrm{T}\}]Y_{++}\exp\{it\Omega\},$$

and, finally,

$$\begin{aligned} \tilde{Y}_{--}(t) &= 'Y_{++} \exp\{it\mathrm{T}\}[1 - \bar{E}_{\mathrm{T}}(t)E]Y_{--} \exp\{-it\Omega\}, \\ \tilde{Y}_{-+}(t) &= 'Y_{++} \exp\{it\mathrm{T}\}[\bar{E} - \bar{E}_{\mathrm{T}}(t)]Y_{++} \exp\{it\Omega\} \end{aligned} \tag{27.23}$$

when we set

$$E_{\mathrm{T}}(t) = \exp\{-it\mathrm{T}\}\bar{E}\exp\{-it\mathrm{T}\}. \tag{27.24}$$

For the operator $\tilde{G}(t)$, which determines the creation factor in (27.21), we find by (24.32) the expression $\tilde{G}(t) = \tilde{Y}_{--}^{-1}(t)\tilde{Y}_{-+}(t)$ or

$$\tilde{G}(t) = \exp\{it\Omega\}Y_{--}^{-1}[1 - \bar{E}_{\mathrm{T}}(t)E]^{-1}[\bar{E} - \bar{E}_{\mathrm{T}}(t)]Y_{++}\exp\{it\Omega\}. \tag{27.25}$$

For the analysis of the time variation of the vacuum state we also need the number $\tilde{\eta}(t) = \exp\{\mp\frac{1}{2}\, Tr\, \tilde{F}(t)\}$, or rather the number

$$|\tilde{\eta}(t)| = \exp\{\mp\tfrac{1}{2}\, Tr \log '\tilde{Y}_{++}(t)\tilde{Y}_{--}(t)\}, \tag{27.26}$$

cf. (24.45). We first note the relation

$$'\tilde{Y}_{++}(t)\tilde{Y}_{--}(t) = \exp\{it\Omega\}\tilde{Y}_{++}[1 - \bar{E}E_T(t)]\exp\{-itT\}Y_{--}'Y_{++}\exp\{itT\} \cdot [1 - \bar{E}_T(t)E]Y_{--}\exp\{-it\Omega\}$$

and make use of the trace composition rule (24.42). We then find

$$\exp Tr \log '\tilde{Y}_{++}(t)\tilde{Y}_{--}(t) = \exp 2\, Tr \log 'Y_{++}Y_{--} \exp 2\, \mathcal{R}e\, Tr \log[1 - \bar{E}_T(t)E].$$

Introducing the number $|\eta|$ given by (24.45) we finally obtain

$$(27.27) \qquad |\tilde{\eta}(t)| = |\eta|^2 \exp\{\mp\tfrac{1}{2}\mathcal{R}e\, Tr \log[1 - \bar{E}_T(t)E]\}.$$

Since $1 - \bar{E}E = (Y_{--}'Y_{++})^{-1}$, cf. (24.39), we have

$$Tr \log 'Y_{++}Y_{--} = Tr \log Y_{--}'Y_{++} = -Tr \log[1 - \bar{E}E]$$

and hence, again by virtue of the trace composition rule, we can write

$$(27.28) \qquad |\eta(t)| = \exp\{\mp\tfrac{1}{2}\mathcal{R}e\, Tr \log[1 - \bar{E}E]^{-1}[1 - \bar{E}_T(t)E]\}.$$

Since $\bar{E}_T(0) = \bar{E}$, it is clear from this expression that

$$(27.29) \qquad |\eta(0)| = 1.$$

The question arises: are there any cases in which *the canonical transformation T* (connecting the modified and unmodified representations) *fails to exist although the transition operator $\tilde{T}(t)$* (connecting unmodified representations at different times) *does exist*. We recall that a similar situation could arise in the case of the infrared catastrophe treated in Part IV. In that case the field was myriotic with respect to the modified annihilation and creation operators B^* while it remained ordinary with respect to the unmodified operators A^* at all times.

A closer analysis seems to indicate that *such a situation cannot arise in the present case*. For, it seems that the trace of $\mathcal{R}e\,[1 - \bar{E}E]^{-1}[\bar{E}E - \bar{E}_T(t)E]$ cannot exist if the trace of $\bar{E}E$ does not exist, although the trace of $\mathcal{R}e\,[\bar{E}E - \bar{E}_T(t)E]$ might exist. Thus it is indicated that *a field which is myriotic with respect to the modified operators B^* becomes myriotic with respect to A^* instantaneously for $t > 0$ even if it was amyriotic with respect to A^* at the time $t = 0$.*

Time Variation of the Vacuum State

Suppose the state Φ is an unmodified vacuum state $\Phi = \Phi_{vac}$ at the time $t = 0$; then the nature of this state at a time $t > 0$ can be read off from the A-representation of the adjusted interaction state defined by (27.12),

$$(27.30) \qquad \tilde{\Phi}_v(t) = \tilde{\eta}(t)\exp[\tilde{\mathfrak{S}}(t)]\Phi_{vac}.$$

For, as remarked in connection with relation (27.70), the $A(0)$-representation of $\tilde{\Phi}(t)$ is the $\tilde{A}(t)$-representation of the state Φ. Hence, according to formula (24.26), the number $|\tilde{\eta}(t)|^2$ is *the probability that the field will be in the vacuum state* *at the time t if it was in the vacuum state at the time $t = 0$.* From (27.27) we find the expression

(27.31) $$P(N(t), 0 \mid N(0), 0) = \mid \eta \mid^4 \exp \{\mp \Re e\, Tr \log [1 - \bar{E}_T(t)E]^{-1}\}.$$

for this probability.

Asymptotic Transition Probabilities

It is not difficult to calculate the limit for $t \to \infty$ of the probability that the field will be in a vacuum state at the time t if it was in such a state at the time $t = 0$, provided the field extends over the infinite space. By making this assumption we exclude, for example, those cases of several interacting media described in Section 22, in which one medium is concentrated at a single point. We may conclude from this assumption that the operator $\bar{E}_T(t)E$ approaches zero as $\mid t \mid \to \infty$. For, in a representation in which the modified energy T is diagonalized, the kernel of the operator $\bar{E}_T(t)E$ is

$$\exp \{itv'\} \int \bar{E}(v', v) \exp \{itv\} E(v , v'') \, dv$$

if $E(v', v'')$ is the kernel of E; here we omit reference to the accessory variables which, together with T, form a complete system of observables. Since $E(v', v'')$ is assumed to be quadratically integrable, it is clear that the integral approaches zero as $\mid t \mid \to \infty$; i.e.,

(27.32) $$\bar{E}_T(t)E \to 0 \quad \text{as} \quad \mid t \mid \to \infty .$$

We recall that a similar argument was used in Part III, cf. (13.45). It follows that $\log [1 - \bar{E}_T(t) E]$ approaches zero, whence relation (27.27) gives

(27.33) $$\mid \eta(t) \mid \to \mid \eta \mid^2$$

or

(27.34)$_0$ $$P(N_\infty , 0 \mid N(0), 0) = \mid \eta \mid^4.$$

In other words, *the probability that the state will eventually be a vacuum state if it was a vacuum state originally is the square of the probability that it will be found to be a modified vacuum state*, cf. (24.26).

This result is a special case of the formula

(27.34) $$P(N_\infty , n \mid N(0), n_0) = \sum_m P(N_\infty , n \mid M, m)P(M, m \mid N(0), n_0)$$

which could be interpreted by saying that *the classical rule of composition of probabilities holds even if the number M of modified particles is not measured after the initial measurement of the number N of unmodified particles.* A similar result was obtained in Part III, cf. (15.40). The probability $P(N_\infty , n \mid M, m)$ in formula (27.34), the limit of $P(N(t), n \mid M, m)$ as $t \to \infty$, can simply be expressed as[9]

(27.35) $$P(N_\infty , n \mid M, m) = P(N, m - n \mid M, 0)$$

[9]Formula (15.40) in Part III should also have been supplemented by formula (27.34). The value of $P(N, m - n \mid M, 0)$ there would be given by (14.55).

and the probability $P(N, m - n \mid M, n)$ is given by formula $(24.26)_1$. We shall not give a proof of formulas (27.34), (27.35) although this could be done easily.

Scattering Operator

For single particles with the energy Υ one may express the Schrödinger state $\Psi_s(t)$ at the time t in terms of the Schrödinger state $\Psi = \Psi_s(0)$ at the time $t = 0$ by the formula

(27.36) $$\Psi_s(t) = \exp\{-it\Upsilon\}\Psi.$$

Considering the operator Υ as a modification of the undisturbed operator Ω one may introduce the interaction state

(27.37) $$\Psi_s(t) = U(t)\Psi$$

with

(27.38) $$U(t) = \exp\{it\Omega\} \exp\{-it\Upsilon\}.$$

Under appropriate conditions on the disturbance $\Upsilon - \Omega$, see, e.g. [45], the operator $U(t)$ approaches limits $U_\pm$ as $t \to \pm\infty$,

(27.39) $$U(t) \to U_\pm \quad \text{as} \quad t \to \pm\infty.$$

The operators $U_\pm$, which may be called "half scattering operators," satisfy the relation, cf. [44] and [45],

(27.40) $$\Omega U_\pm = U_\pm \Upsilon.$$

The scattering operator, which we here denote by U_{+-}, may then be defined as

(27.41) $$U_{+-} = U_+ U_-^{-1};$$

it evidently commutes with the undisturbed energy operator Ω,

(27.42) $$\Omega U_{+-} = U_{+-}\Omega.$$

Of course, the scattering operator may also be defined as the limit of the "incomplete scattering operator"

(27.42) $$U_{+-}(t, t_0) = U(t)U^{-1}(t_0) = \exp\{it\Omega\} \exp\{Ui(t_0 - t)\Upsilon\} \exp\{-it_0\Omega\},$$

i.e.

$(27.39)_{+-}$ $$U_{+-}(t, t_0) \to U_{+-} \quad \text{as} \quad t \to \infty,\ t_0 \to -\infty.$$

The analogue of the operator $U(t)$ for the fields under consideration is the operator $\tilde{T}^{-1}(t)$ as seen from (27.13); the analogue of the operator $U_{+-}(t, t_0) = U(t)U^{-1}(t_0)$ is the operator

(27.44) $$T_{+-}^{-1}(t, t_0) = \tilde{T}^{-1}(t)\tilde{T}(t_0),$$

which we also term "incomplete scattering operator." Neither the operators $\tilde{T}(t)$ nor the operators $T_{+-}^{-1}(t, t_0)$ approach limits as $t \to +\infty$, $t_0 \to -\infty$. For

this reason we must be satisfied with an asymptotic description of these operators for large values of t. This description will be given only for $T^{-1}_{\cdot\cdot}(t, t_0)$.
The formula

$$T_{\cdot\cdot}(t, t_0) = \exp\{it_0[\hat{\Omega}]\}\tilde{T}(t - t_0) \exp\{-it_0[\hat{\Omega}]\}, \tag{27.45}$$

which can be deduced from formulas (27.10), will be useful for this purpose.

Scattering Operator According to Yang and Feldman

There is another possibility of introducing the scattering operator. From formula (27.18) we see that the pseudo-operator $\tilde{\mathfrak{a}}(t)$ can be expressed in terms of the pseudo-operator $\mathfrak{a}(t_0)$ by the relation

$$\tilde{\mathfrak{a}}(t) = \tilde{\mathcal{Y}}(t, t_0)\tilde{\mathfrak{a}}(t_0)$$

in which the linear transformation

$$\tilde{\mathcal{Y}}(t, t_0) = {}'\tilde{\mathcal{Y}}(t)\tilde{\mathcal{Y}}(t_0)$$

again satisfies the condition $\tilde{\mathcal{Y}}(t, t_0)'\tilde{\mathcal{Y}}(t, t_0) = 1$. Relation (27.20) leads to the relation

$$\mathfrak{a}(t) = S^{-1}(t, t_0)\mathfrak{a}(t_0)S(t, t_0)$$

in which

$$S(t, t_0) = \tilde{T}(t_0)\tilde{T}^{-1}(t).$$

This operator is therefore the canonical transformation associated with the linear transformation $\tilde{\mathcal{Y}}(t, t_0)$.[10] If the operator $S(t, t_0)$ approached a limit as $t_0 \to -\infty$, $t \to \infty$, this limit would be the scattering operator in the sense of Yang and Feldman, [5].

From the definition of the operator $S(t, t_0)$ it is clear *that the $\tilde{\mathfrak{a}}(t_0)$-representation of the state $S(t, t_0)\,\Phi$ is a $\tilde{\mathfrak{a}}(t)$-representation of the Heisenberg state Φ.*

The connection between the operator $S(t, t_0)$ and the operator $\tilde{T}^{-1}_{\cdot\cdot}(t, t_0)$ introduced above is evidently given by the relation

$$S(t, t_0) = \tilde{T}(t_0)\tilde{T}^{-1}_{\cdot\cdot}(t, t_0)\tilde{T}^{-1}(t_0),$$

in agreement with the fact that the operator $\tilde{T}^{-1}_{\cdot\cdot}(t, t_0)$ transforms the interaction state $\tilde{T}^{-1}(t_0)\,\Phi$ into the interaction state $\tilde{T}^{-1}(t)\,\Phi$.

This relation has the following consequence. Suppose we express the operator $\tilde{T}^{-1}_{\cdot\cdot}(t, t_0)$ in terms of the operators $A^*(0)$ as

$$\tilde{T}^{-1}_{\cdot\cdot}(t, t_0) = f(A^*(0)),$$

then, by virtue of (27.20),

$$S(t, t_0) = f(\tilde{A}^*(t_0)).$$

[10] This connection was pointed out to me by B. Zumino. It will be the starting point for his forthcoming paper.

Therefore, in describing the effect of the operator $\tilde{T}^{-1}_{**}(t, t_0)$ in terms of the $A(0)$-representers of a state we shall have described the effect of the operator $S(t, t_0)$ in terms of the $\tilde{A}(t)$-representers of a state.

Asymptotic Field Scattering Operator

We proceed to show that for large positive values of t and $-t_0$ the operator $T^{-1}_{**}(t, t_0)$ admits the description

$$T^{-1}_{**}(t, t_0) = \exp\{-\tfrac{1}{2}A^+G(-t)A^+\}\exp\{\tfrac{1}{2}A^+G_+(-t_0)A^+\} \cdot \exp\{\tfrac{1}{2}i(t-t_0)h\}\exp\{A^+\log U_{**}A^-\} \tag{27.46}$$

with

$$G(-t) = \exp\{it\Omega\}G\exp\{it\Omega\} \tag{27.47}$$

and

$$G_+(-t_0) = \exp\{it_0\Omega\}U_+\bar{E}'U_+\exp\{it_0\Omega\}. \tag{27.48}$$

Here U_{**} is the scattering operator and U_+ the half scattering operator for a single particle, cf. (27.39) (27.41). In deriving formula (27.46) we shall make two assumptions. *Assumption I: the field extends over the whole space. Assumption II: the operators Y_{--} and Y_{++} differ from the identity by a square traceable operator.* The latter assumption is satisfied in the problem of the boson field treated in Sections 22 and 26. We shall first justify formula (24.47) and then give an interpretation of it.

Justification

We employ the second form of the operator $T_{**}(t, t_0)$ given by (24.16) and write it as

$$T_{**}(t, t_0) = \exp\{[\mathcal{E}_{**}(t, t_0)]\}\exp\{[\mathcal{F}_{**}(t, t_0)]\}\exp\{[\mathcal{G}_{**}(t, t_0)]\}. \tag{27.49}$$

Using the form (27.45) of $T_{**}(t, t_0)$ and observing (24.30), (24.31) we obtain

$$\begin{aligned} E_{**}(t, t_0) &= \exp\{-it_0\Omega\}\tilde{E}(t-t_0)\exp\{-it_0\Omega\}, \\ G_{**}(t, t_0) &= \exp\{it_0\Omega\}\tilde{G}(t-t_0)\exp\{it_0\Omega\}, \\ \exp F_{**}(t, t_0) &= \exp\{it_0\Omega\}\tilde{Y}_{--}(t-t_0)\exp\{-it_0\Omega\}. \end{aligned} \tag{27.50}$$

From (27.25) we then have

$$G_{**}(t, t_0) = \exp\{it\Omega\}Y^{-1}_{--}[1-\bar{E}_T(t-t_0)E]^{-1} \cdot [\bar{E}-\bar{E}_T(t-t_0)]Y_{++}\exp\{it\Omega\}; \tag{27.51}$$

from (27.23) we have

$$\exp F_{**}(t, t_0) = \exp\{it_0\Omega\}'Y_{++}\exp\{i(t-t_0)\mathrm{T}\} \cdot [1-\bar{E}_T(t-t_0)E]Y_{--}\exp\{-it\Omega\}, \tag{27.52}$$

and $E_{..}(t, t_0)$ is given by a formula similar to (27.50).

As a consequence of Assumption I we have $\bar{E}_T(t)E \to 0$ as before, cf. (27.32). Arguments similar to those used to derive (27.32) yield the relation

$$\exp \{[\mathcal{E}]\} \exp \{itA^+\Omega A^-\}\Phi \to \Phi \quad \text{as} \quad |t| \to \infty. \tag{27.53}$$

More specifically this formula can be derived from the fact that

$$\iint E(s_1, s_2) \exp \{it\Omega^{\cdot\cdot}\} \exp \{it\Omega^{\cdot\cdot}\}\psi(s_1, s_2)\, dm(s_1)\, dm(s_2) \to 0 \quad \text{as} \quad |t| \to 0$$

for every function $\psi(s_1, s)$ with $\iint |\psi(s_1, s_2)|^2\, dm(s_1)\, dm(s_2) < \infty$. From the form (27.50) of the operator $E_{..}(t)$ one may deduce by similar arguments the relation

$$\exp \{-[\mathcal{E}_{..}(t, t_0)]\}\bar{\Psi} \to \bar{\Psi} \quad \text{as} \quad |t| \to \infty. \tag{27.54}$$

In other words, *the annihilation operators do not contribute asymptotically.*

In order to determine the asymptotic contribution from $\exp \{[F_{..}(t)]\}$ we make use of Assumption II and deduce from it the relations

$$\begin{aligned} &\exp \{it\Omega\} Y_{--} \exp \{-it\Omega\} \to 1 \\ &\exp \{-it\Omega\}' Y_{++} \exp \{it\Omega\} \to 1 \quad \text{as} \quad |t| \to \infty. \end{aligned} \tag{27.55}$$

From (27.52) we see that the operator $\exp F_{..}(t)$ is given asymptotically by

$$\exp F_{..}(t, t_0) \sim \exp \{it_0\Omega\} \exp \{i(t - t_0)\Upsilon\} \exp \{-it\Omega\}$$

or, because of (24.44), by

$$\exp F_{..}(t, t_0) \sim U_{..}^{-1}(t, t_0) \quad \text{as} \quad t \sim \infty,\ t \sim -\infty. \tag{27.56}$$

Consequently, because of (27.09), we have

$$\begin{aligned} \exp \{[F_{..}(t, t_0)]\} &\sim \exp \{-[\log U_{..}(t, t_0)]\} \\ &= \exp \{-\tfrac{1}{2}\, Tr \log U_{..}(t, t_0)\} \exp \{-A^+ \log U_{..}(t, t_0) A^-\}. \end{aligned}$$

Now,

$$Tr \log U_{..}(t, t_0) = -i(t - t_0)\, Tr\, (\Upsilon - \Omega) = -i(t - t_0)h$$

as seen from (27.43). Hence, because of $(27.39)_{..}$, relation

$$\exp \{[F_{..}(t, t_0)]\} \sim \exp \{\tfrac{1}{2}i(t - t_0)h\} \exp \{-A^+ \log U_{..} A^-\} \tag{27.57}$$

follows.

From relation (27.51) we finally have

$$\begin{aligned} G_{..}(t, t_0) \sim{}& \exp \{it\Omega\} Y_{--}^{-1} \bar{E} Y_{++} \exp \{it\Omega\} \\ &- \exp \{it\Omega\} Y_{--}^{-1} \exp \{-i(t - t_0)\Upsilon\} \bar{E} \exp \{-i(t - t_0)\Upsilon\} Y_{++} \exp \{it\Omega\}. \end{aligned}$$

Now, $Y_{--}^{-1}\bar{E}Y_{++} = G$ by (24.32), (24.35); hence, by (27.55), (27.38), (27.39) and (27.47), (27.48)

(27.58) $$G_{..}(t, t_0) \sim G(-t) - G_+(-t_0).$$

In virtue of $\exp(\mathcal{G}_1 + \mathcal{G}_2) = \exp \mathcal{G}_1 \exp \mathcal{G}_2$ and $[\mathcal{G}] = \frac{1}{2}A^+GA^+$ formula (27.46) follows

Interpretation

The operator $\exp\{A^+ \log U_{..}A^-\}$ which occurs in formula (24.47) will, somewhat improperly, be called the *biquantized scattering operator.* For, this operator transforms the state Φ_*, represented as

(27.60) $$\Phi_* \leftrightarrow \{\psi_n(s)_n\}$$

into the state which is represented as

(27.60)$_1$ $$\exp\{A^+ \log U_{..}A^-\}\Phi_* \leftrightarrow \{U_{..}^{..} \cdots U_{..}^{..}\psi_n(s)_n\};$$

thus *the representers of this state are obtained by letting the operator $U_{..}$ act on the representers $\psi_n(s_1, \cdots, s_n)$ considered successively as functions of $s_1, \cdots, s_n$.*

In order to describe the nature of the operator $\exp\{\frac{1}{2}A^+G(-t)A^+\}$, which also occurs in formula (24.42), we first observe that it transforms the vacuum state Φ_{vac} into the state represented as

(27.61) $$\exp\{\tfrac{1}{2}A^+G(-t)A^+\}\Phi_{vac} \leftrightarrow \left\{(n!)^{1/2}\begin{pmatrix}\text{Sy}\\ \text{Asy}\end{pmatrix}g(s_1, s_2, -t) \cdots g(s_{n-1}, s_n, -t)\right\}.$$

Here

(27.62) $$g(s_1, s_2, -t) = \exp\{it\Omega^{\cdot\cdot}\}\exp\{it\Omega^{\cdot\cdot}\}g(s_1, s_2)$$

is the kernel of the integral operator $G^{\cdot}(-t)$. Similarly, the operator $\exp\{\frac{1}{2}A^+G_+(-t_0)A^+\}$ transforms the vacuum state into the state represented as

(27.63) $$\exp\{\tfrac{1}{2}A^+ G_+(-t_0)A^+\}\Phi_{vac} \leftrightarrow \left\{(n!)^{1/2}\begin{pmatrix}\text{Sy}\\ \text{Asy}\end{pmatrix}g_+(s_1, s_2, -t_0) \cdots g_+(s_{n-1}, s_n, -t)\right\}$$

with

(27.64) $$g_+(s_1, s_2, -t_0) = \exp\{it_0\Omega^{\cdot\cdot}\}\exp\{it_0\Omega^{\cdot\cdot}\}U_+^{\cdot\cdot}U_+^{\cdot\cdot}\bar{e}(s_1, s_2).$$

Here $\bar{e}(s_1, s_2)$ is the kernel of the integral operator $\bar{E}^{\cdot}$.

Combining the effects of the operators $\exp\{A^+\log U_{..}A^-\}$, $\exp\{\frac{1}{2}A^+G(-t)A^+\}$ and $\exp\{\frac{1}{2}A^+G_+(-t)A^+\}$ we finally see from (27.59) that the state $T_{..}^{-1}(t, t_0)\,\Phi$ is represented as

$$
(27.65)\qquad
\begin{aligned}
\Phi_w(t) = T^{-1}_{ss}(t, t_0)\Phi_w'(t_0) \leftrightarrow \eta \exp\{\tfrac{1}{2}i(t - t_0)h\} \\
\cdot\Big\{\sum_{n_0} \sum_{n_1=n_0}^{n} (n!n_0!)^{1/2}2^{n_0-n}\binom{\mathrm{Sy}}{\mathrm{Asy}} g(s_n, s_{n-1}, -t) \\
\cdots g(s_{n_1+2}, s_{n_1+1}, -t)g_+(s_{n_1}, s_{n_1-1}, -t_0) \\
\cdots g_+(s_{n_0+2}, s_{n_0+1}, -t_0)U^{s n_0}_{ss} \cdots U^{s 1}_{ss}\psi_{n_0}(s)_{n_0}\Big\}.
\end{aligned}
$$

This formula can be interpreted somewhat more easily if we introduce the Schrödinger states

$$(27.66)\qquad \Phi_s(t) = \exp\{-itA^+\Omega A^-\}\Phi_w(t)$$

and

$$(27.66)_0\qquad \Phi_s(t_0) = \exp\{-it_0A^+\Omega A^-\}\Phi_w(t_0).$$

The state (27.67) has the A_s-representation

$$(27.67)\qquad \Phi_s(t_0) \leftrightarrow \{\exp\{-it_0\Omega^{s_1}\} \cdots \exp\{-it_0\Omega^{s_n}\}\psi_n(s)_n\},$$

while the representation of (27.66) is derived from (27.65), (27.64), (27.62) as

$$(27.68)\qquad \Phi_s(t) \leftrightarrow \{\psi_n(s)_n(t)\},$$

with

$$
(27.69)\qquad
\begin{aligned}
\psi_n(s)_n(t) = \eta \exp\left\{\frac{1}{2}i(t - t_0)h\right\} \sum_{n_0} \sum_{n_1=n_0}^{n} (n!n_0!)^{1/2}2^{n_0-n} \\
\cdot\binom{\mathrm{Sy}}{\mathrm{Asy}} g(s_n, s_{n-1}) \cdots g(s_{n_1+2}, s_{n_1+1}) \exp\{i(t_0 - t)(\Omega^{n_1} + \Omega^{n_1-1})\} \\
\cdot g_+(s_{n_1}, s_{n_1-1}) \cdots \exp\{i(t_0 - t)(\Omega^{n_0+2} + \Omega^{n_0+1})\} g(s_{n_0+2}, s_{n_0+1}) \\
\cdot \exp\{-it\Omega^{n_0}\}U^{n_0}_{ss} \cdots \exp\{-it\Omega^{1}\}U^{1}_{ss}\psi_{n_0}(s)_{n_0}.
\end{aligned}
$$

Here the abbreviation $n_0, \cdots, 1, \cdots, n_1, \cdots, n_0 + 1$ has been used for the superscripts $s_{n_0}, \cdots, s_1, \cdots, s_{n_1}, \cdots, s_{n_0+1}$.

The functions $\psi_n(s)_n(t)$ given by (27.69) *are the* $A(t)$*-representers of the state* Φ since they are the $A(0)$-representers of the Schrödinger state $\Phi_s(t)$.

In order to interpret formula (27.69), we suppose that n_0 particles were present in the remote past and describe the amplitude of the probability of finding n particles at some time in the future as being composed of amplitudes which correspond to the following processes: *A certain number,* $n - n_1$, *of* *particles are spontaneously emitted at the final time, the n_0 original particles are still present but have been scattered, and the other $n_1 - n_0$ particles were emitted spontaneously at the original time and have then undergone a half-scattering process.*

This asymptotic description of the field scattering suffers from a certain incongruity. In deriving formula (27.47) we assumed that the operator $T^{-1}_{..}(t, t_0)$ was to be applied on the interaction state $\Phi_w(t_0)$ at the time t_0. If this state $\Phi_w(t_0)$ approached a limit state as $t \rightarrow \infty$ (as it usually does in single particle scattering problems) it would be justified to substitute this limit state for the state $\Phi_w(t_0)$. In the present problem, however, no such limit state exists.

Nevertheless, in one typical situation, namely, when the disturbance Q is absent up to the initial time and is only switched on at this time, the asymptotic description (24.47) can be used, provided the particles composing the field in the initial state are far removed from the region in which the disturbance is noticeable. The arguments that led to the simplified form (24.47) of the transition amplitudes then remain valid.

From the fact that the amplitudes of the three different types of particles occurring in formula (27.69) have different time-dependent phase factors, $\exp\{-i(t - t_0)\Omega\}$, $\exp\{-it\Omega\}$, 1, one can deduce: *At the final time those particles are farthest removed from the disturbance region which were spontaneously emitted originally and then half scattered; next come the fully scattered original particles, while the unscattered spontaneously emitted particles are not separated from the disturbance region.*

Another consequence of the difference of these phase factors is that the contributions of the three types of particles to the *transition probabilities* are asymptotically independent of each other as $t \rightarrow \infty$. In other words, the classical rule of composition of probabilities holds asymptotically. Although it could easily be done, we shall not calculate these asymptotic transition probabilities.

28. *A Modified Electron-Positron Field*

The general theory developed in the preceding sections is valid for fermion as well as boson fields, but has so far been applied only to boson fields. As an application to fermion fields we now consider a field of electrons and positrons in the sense of Dirac under the influence of external forces which cause a linear homogeneous transformation of the annihilation and creation operators. Electromagnetic forces produced by an unquantized electromagnetic field have this property. The modification of the electron-positron field caused by such forces includes as a special case the "polarization of the vacuum." Our theory would allow us to determine this modification explicitly if the external forces satisfied the severe conditions under which the canonical transformation exists. These conditions, however, are satisfied only for special external electromagnetic fields, for example, for time-independent purely electric fields. Thus it is clear that the infinities which have been found in the investigation of the vacuum polarization are not just caused by the perturbation approximation but are inherent in the problem.

Dirac Electron

To describe the field of electrons and positrons by the process of biquantisation we have first to describe *the operators formerly associated with a single Dirac electron.*

Dirac's theory of the electron was developed in connection with the requirements of the principle of relativity. The customary presentations of this theory naturally emphasize this connection; it will, however, not be apparent in the subsequent treatment. We shall select a definite Lorentz frame and employ a complete set of observables which depends on this frame. Moreover, we shall confine ourselves to external forces which do not depend on the time defined with reference to the selected frame. In principle, our procedure could be carried out also for external forces which do depend on the time (see the remarks made at the end of the subsection on "Direct Method" in Section 27); for, in this case it should be possible to cast the results into a form which exhibits invariance under Lorentz transformation. We shall not attempt to pursue this idea in the present exposition. (See a forthcoming paper by B. Zumino.)

In the following we shall deviate from our custom of denoting operators by capital letters and their eigenvalues by corresponding lower case letters. The basic single particle operators will be denoted by small letters while their eigenvalues will be indicated by primes. Thus we shall come closer to the customary notation.

An important feature of Dirac's electron is that the three components x_1, x_2, x_3 of the position x do not form a complete set of observables sufficient for its description. Additional observables are needed. Among possible additional observables one should first mention the three components of the spin $\frac{1}{2}\hbar\sigma_1$, $\frac{1}{2}\hbar\sigma_2$, $\frac{1}{2}\hbar\sigma_3$. We prefer to work with the *spin signatures* σ_1, σ_2, σ_3 and—as before—set $\hbar = 1$. The corresponding operators satisfy the relations

$$\sigma_\lambda\sigma_\nu + \sigma_\nu\sigma_\lambda = \begin{cases} 2\delta_{\lambda\nu} = 2 & \text{if} \quad \lambda = \nu \\ 0 & \text{if} \quad \lambda \neq \nu \end{cases} \tag{28.1}$$

and

$$\sigma_1\sigma_2 = i\sigma_3, \qquad \sigma_2\sigma_3 = i\sigma_1, \qquad \sigma_3\sigma_1 = i\sigma_2. \tag{28.2}$$

The triple $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ transforms like a vector when $x = (x_1, x_2, x_3)$ undergoes a rotation. There is a second set of observables β_1, β_2, β_3 which satisfy similar relations, but are not affected by a rotation of x. It is stipulated that the observables σ commute with β_1, β_2, β_3 and that all of them commute with x and p. It is further stipulated that x together with one of the variables σ and one of the variables β form a complete set.

The energy operator proposed by Dirac can be written in the form

$$\begin{aligned} H_0 &= \beta_1(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3) + \mu\beta_3 \\ &= \beta_1(\sigma p) + \mu\beta_3 \end{aligned} \tag{28.3}$$

in which $p = (p_1, p_2, p_3)$—instead of k—is the momentum, μ the rest mass, and β a particular one among the observables $\beta_1, \beta_2, \beta_3$, say $\beta = \beta_3$. The observables $\beta_1\sigma_1, \beta_1\sigma_2, \beta_1\sigma_3$ are customarily denoted by $\alpha_1, \alpha_2, \alpha_3$.

As the representing observables we select the position x, the observable β, and—as customary—the spin signature σ_3. Their eigenvalues, the "quantum variables", will be denoted by x', β', σ'—without subscript. The representer of a state Ψ of the electron will be denoted by ψ, so that

$$\Psi \leftrightarrow \psi(x', \sigma', \beta'). \tag{28.4}$$

The unit form is

$$(\Psi, \Psi) = \sum_{\sigma'\beta'} \int | \psi(x', \sigma', \beta') |^2 \, dx'. \tag{28.5}$$

The eigenvalues of σ_3 and β are $\sigma' = \pm 1, \beta' = \pm 1$, and each value is assumed once. Hence we may also write

$$\Psi \leftrightarrow \{\psi(x', 1, 1), \psi(x', 1, -1), \psi(x', -1, 1), \psi(x', -1, -1)\} \tag{28.4$'$}$$

and

$$\begin{aligned}(\Psi, \Psi) = \int [| \psi(x', 1, 1) |^2 &+ | \psi(x', 1, -1) |^2 \\ &+ | \psi(x', -1, 1) |^2 + | \psi(x', -1, -1) |^2] \, dx'.\end{aligned} \tag{28.5$'$}$$

It is customary to use a subscript running from 1 to 4 instead of the four pairs (1, 1), (1, −1), (−1, 1), (−1, −1), but the latter notation seems somewhat more suitable for our purposes. Aside from this point our treatment is similar to that of Dirac [2].

The representation (28.4) can be so chosen that the matrix elements of $\sigma_1, \sigma_2, \sigma_3, \beta, \beta_1$ are

$$\begin{aligned}(\sigma' \mid \sigma_1 \mid \sigma'') &= \frac{1}{2}(1 - \sigma'\sigma'') \\ (\sigma' \mid \sigma_2 \mid \sigma'') &= \frac{i}{2}(\sigma' - \sigma'') \\ (\sigma' \mid \sigma_3 \mid \sigma'') &= \frac{1}{2}(\sigma' + \sigma''),\end{aligned} \tag{28.6}$$

$$\begin{aligned}(\beta' \mid \beta \mid \beta'') &= \frac{1}{2}(\beta' + \beta'') \\ (\beta' \mid \beta_1 \mid \beta'') &= \frac{1}{2}(1 - \beta'\beta'').\end{aligned} \tag{28.7}$$

This description is, of course, equivalent to the more customary description

$$(28.6)' \qquad \sigma_1 \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_3 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$(28.7)' \qquad \beta \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta_1 \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the x-representer of the momentum is $-i\,\nabla_x$, the (x, β, σ_3)-representation of the state $H\,\Psi$ is

$$(28.8) \qquad \begin{aligned} H_0\Psi &\leftrightarrow H_0^{x,\sigma_3,\beta}\psi(x', \sigma', \beta') \\ &= \sum_{\beta'',\sigma''} [-i(\beta' \mid \beta_1 \mid \beta'')(\sigma' \mid \sigma\nabla_{x'} \mid \sigma'') + \mu(\beta' \mid \beta \mid \beta'')]\psi(x', \sigma'', \beta''). \end{aligned}$$

The potential V of an external force will in general be a function of x and of $\alpha = \beta_1\,\sigma$. The canonical transformation, however, will not exist for all such forces, and we therefore assume a more general type of force with a potential whose representer is an integral operator with a sufficiently smooth kernel. The representer of this kernel will be denoted by $V^{x,\sigma_3,\beta}$. The modified energy operator is then

$$(28.9) \qquad H_{mod} = H_0 + V$$

and its representer

$$(28.9)' \qquad H_{mod}^{x,\sigma_3,\beta} = H_0^{x,\sigma_3,\beta} + V^{x,\sigma_3,\beta}.$$

Transformation of the Quantum Variables

Instead of the position x we may introduce the momentum p by applying the Fourier transformation to the representer $\psi(x', \sigma', \beta')$. Thus p, σ_3, β form a complete commuting system. Instead of σ_3 we may introduce the variable

$$(28.10) \qquad \tau = (\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3) \mid p \mid^{-1} = (\sigma p) \mid p \mid^{-1},$$

the component of the spin signature σ in the direction of the momentum p. It is immediately seen that this observable has also the eigenvalues $\tau' = \pm 1$, each assumed once. Thus p, τ, β form a complete commuting set. Note that the undisturbed energy H_0 can be written in the form

$$(28.11) \qquad H_0 = \tau \mid p \mid \beta_1 + \mu\beta$$

and since $\tau^2 = \beta_1^2 = \beta^2 = 1$ we see that

$$H_0^2 = \mid p \mid^2 + \mu^2.$$

Introducing the "absolute undisturbed energy"

$$(28.12) \qquad \Omega = [\mid p \mid^2 + \mu^2]^{1/2}$$

and the "sign of the undisturbed energy"

$$\epsilon = H_0\Omega^{-1}, \tag{28.13}$$

we may write H_0 in the form

$$H_0 = \epsilon\Omega. \tag{28.14}$$

Clearly, the eigenvalues of ϵ are $\epsilon' = \pm 1$, each being assumed once. The undesirable feature that the energy may be in a state with a negative energy can be eliminated by considering a field composed of electrons and positrons, as will be discussed below.

We may introduce the sign of the energy ϵ as observable instead of β and thus represent the state Ψ by a function $\phi(p', \tau', \epsilon')$,

$$\Psi \leftrightarrow \phi(p', \tau', \epsilon') \tag{28.15}$$

such that

$$(\Psi, \Psi) = \sum_{\epsilon', \tau'} \int | \phi(p', \tau', \epsilon') |^2 \, dp'. \tag{28.16}$$

The transformation of the representers ϕ into the representers ψ could be given explicitly, cf. e.g. [43]; it is of the form

$$\psi(x', \sigma', \beta') = \sum_{\epsilon', \tau'} \int (4\pi)^{-3/2} \exp \{ix'p'\} \cdot (p', \sigma', \beta' \mid p', \tau', \epsilon')\phi(p', \tau', \epsilon') \, dp' \tag{28.17}$$

with

$$x'p' = x_1'p_1' + x_2'p_2' + x_3'p_3' ,$$

but we shall not use this explicit expression. We are satisfied with writing the transformation (28.17) in the form

$$\psi(x', \sigma', \beta') = \sum_{\epsilon'} R^{p', \tau'}_{x, \sigma, \beta}(\epsilon')\phi(p', \tau', \epsilon') \tag{28.18}$$

in which the terms $R^{p, \tau}_{x, \sigma, \beta}$ (± 1) stand for two transformations which transform functions of p', τ' into functions of x, σ, β. The variable ϵ is treated differently because it will play a special role later on.

Clearly, the transformation (28.18) gives a spectral transformation of the undisturbed energy H_0 ; for, evidently,

$$H_0^{x, \sigma, \beta}\psi(x', \sigma', \beta') = \sum_{\epsilon'} R^{p, \tau}_{x, \sigma, \beta}(\epsilon')\epsilon'\omega'\phi(p', \tau', \epsilon') \tag{28.19}$$

with $\omega' = [\, | p' |^2 + \mu^2]^{1/2}$.

Let us assume that a spectral transformation of the disturbed operator $H_{\text{mod}} = H_0 + V$ can also be constructed. We need not assume that the operator H_{mod} can be obtained from H_0 by a canonical transformation; we need not even assume that the spectrum of H_{mod} is of the same type as that of H_0. We do assume, however, that variables $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$, $\hat{\tau}$, $\hat{\epsilon}$ of the same type as p, τ, ϵ exist such that H_{mod} is of the form

$$H_{mod} = \hat{\epsilon}\Upsilon \tag{28.20}$$

in which Υ is a non-negative operator commuting with $\hat{\epsilon}$. Actually, this assumption is not a restriction; moreover, the variables $\hat{p}$, $\hat{\tau}$ drop out from our final formulas. The assumption was made only to enable us to apply the procedure developed in the preceding sections without modification.

The eigenvalues of the observables $\hat{p}$, $\hat{\tau}$, $\hat{\epsilon}$ will be denoted by p', τ', ϵ' or p'', τ'', ϵ'' without showing the "roof" explicitly. This simplification cannot cause confusion since the eigenvalues of the roofed and unroofed observables run over the same spectrum.

The $(\hat{p}, \hat{\tau}, \hat{\epsilon})$-representers of the state Ψ will be denoted by $\hat{\phi}(p', \tau', \epsilon')$ and the transformation of these representers into $\psi(x, \sigma_3, \beta)$ will be denoted by

$$\psi(x', \sigma_3', \beta') = \sum_{p'\tau'} \hat{R}^{p;\tau}_{x;\sigma_3;\beta}(\epsilon')\hat{\phi}(p', \tau', \epsilon'). \tag{28.21}$$

This transformation need not diagonalize the modified energy $H_0 + V$; it is sufficient that the relation

$$(H_0^{x;\sigma_3;\beta} + V^{x;\sigma_3;\beta})\psi(x, \sigma_3, \beta) = \sum_{p'} \hat{R}^{p;\tau}_{x;\sigma_3;\beta}(\epsilon')\epsilon'\Upsilon^{p;\tau;\tau}\hat{\phi}(p', \tau', \epsilon') \tag{28.22}$$

holds.

The inverse transformation of $R^{p;\tau}_{x;\sigma_3;\beta}(\epsilon')$ will be denoted by

$$R^{x;\sigma_3;\beta}_{p;\tau}(\epsilon')$$

so that

$$R^{x;\sigma_3;\beta}_{p;\tau}(\epsilon')R^{p;\tau}_{x;\sigma_3;\beta}(\epsilon'') = \delta(\epsilon' - \epsilon'') = \tfrac{1}{2}(1 + \epsilon'\epsilon''). \tag{28.23}$$

In the following we actually need only the transformation of the (p, τ, ϵ)-representers into the $(\hat{p}, \hat{\tau}, \hat{\epsilon})$-representers. This transformation will be described as

$$\hat{\phi}(p', \tau', \epsilon') = \sum_{\epsilon''} U^{-}(\epsilon', \epsilon'')\phi(p', \tau', \epsilon''). \tag{28.24}$$

The operators $U^{-}(\pm 1, \pm 1)$ act on functions of p', τ' and produce functions of the same variables. Evidently, these operators are given by

$$U^{-}(\epsilon', \epsilon'') = R^{x;\sigma_3;\beta}_{p;\tau}(\epsilon')R^{p;\tau}_{x;\sigma_3;\beta}(\epsilon''). \tag{28.25}$$

By $U^{+}(\epsilon', \epsilon'')$ we denote the complex conjugate of the operator $U^{-}(\epsilon', \epsilon'')$,

$$U^{+}(\epsilon', \epsilon'') = \overline{U^{-}(\epsilon', \epsilon'')}. \tag{28.26}$$

Since the adjoint of an operator is the conjugate of its Hermitian adjoint, we find

$$'U^{+}(\epsilon', \epsilon'') = R^{x;\sigma_3;\beta}_{p;\tau}(\epsilon'')\hat{R}^{p;\tau}_{x;\sigma_3;\beta}(\epsilon') \tag{28.25'}$$

and

$$'U^{-}(\epsilon', \epsilon'') = \overline{'U^{+}(\epsilon', \epsilon'')}. \tag{28.26$'$}$$

The identities

$$\begin{aligned} \sum_{\epsilon'''} {}'U^{-}(\epsilon''', \epsilon')U^{-}(\epsilon''', \epsilon'') &= \delta(\epsilon' - \epsilon'') \\ \sum_{\epsilon'''} U^{-}(\epsilon', \epsilon''')'U^{-}(\epsilon'', \epsilon''') &= \delta(\epsilon' - \epsilon''). \end{aligned} \tag{28.27}$$

can also be verified.

Electron-Positron Field

The process of biquantization could be performed by introducing a field whose states can be represented in the form

$$\Phi \leftrightarrow \{\phi_n(p, \tau, \epsilon)_n\},$$

by functions $\phi_n(p, \tau, \epsilon)_n$ of the n triples $p_1, \tau_1, \epsilon_1, \cdots, p_n, \tau_n, \epsilon_n$. Here we omit the primes since the variables carry subscripts.

The value of ϕ_n is the amplitude of the probability that there are n particles with the momenta, spin components, and energy signs $p_1, \tau_1, \epsilon_1, \cdots, p_n, \tau_n, \epsilon_n$ respectively. Instead of ϕ_n, one could introduce representers $\psi_n(x, \sigma_3, \beta)_n$ obtained from $\phi_n(p, \tau, \epsilon)_n$ by applying the transformation $R^{p\cdot\tau}_{x\cdot\sigma_3\cdot\beta}(\epsilon)$ on ϕ_n as functions of each of the triples $p_1, \tau_1, \epsilon_1, \cdots, p_n, \tau_n, \epsilon_n$. The corresponding annihilation operators would then be transformed in the same way. Denoting the annihilation operators associated with the (p, τ, ϵ)-representation by $A^{-}_{\epsilon 1}(p', \tau', \epsilon')$ and those associated with the (x, σ_3, β)-representation by $A^{-}_{\epsilon 1}(x', \sigma', \beta')$, we would have

$$A^{-}_{\epsilon 1}(x', \sigma', \beta') = \sum_{\epsilon'} R^{p\cdot\tau}_{x\cdot\sigma_3\cdot\beta}(\epsilon') A^{-}_{\epsilon 1}(p', \tau', \epsilon'). \tag{28.28}$$

The change of these operators in time would then be the same as that of the particle representers,

$$A^{-}_{\epsilon 1}(p', \tau', \epsilon', t) = \exp\{-it\epsilon'\omega'\} A^{-}_{\epsilon 1}(p', \tau', \epsilon') \tag{28.29}$$

with $\omega' = [|p'|^2 + \mu^2]^{1/2}$ and

$$A^{-}_{\epsilon 1}(x', \sigma', \beta', t) = \sum_{\epsilon'} R^{p\cdot\tau}_{x\cdot\sigma_3\cdot\beta}(\epsilon') \exp\{-it\epsilon'\omega'\} A^{-}_{\epsilon 1}(p', \tau', \epsilon'). \tag{28.30}$$

The latter quantity evidently satisfies the differential equation

$$i\nabla_t A^{-}_{\epsilon 1}(x', \sigma', \beta', t) = H_0^{x\cdot\sigma_3\cdot\beta} A^{-}_{\epsilon 1}(x', \sigma', \beta', t). \tag{28.31}$$

Formulas for creation operators would be obtained by taking the complex conjugates of these relations.

Actually, one performs the process of biquantization in a different way. One describes the field as consisting of electrons and positrons, both with the positive energy $\omega = [|p|^2 + \mu^2]^{1/2}$ and represents the states of the field in the form

$$\Phi \leftrightarrow \{\chi_n(p, \tau, \epsilon)_n\} \tag{28.32}$$

in terms of functions χ_n of $p_1, \tau_1, \epsilon_1, \cdots, p_n, \tau_n, \epsilon_n$ whose significance, however, differs from that of the functions ϕ_n. The function $\chi_1(p_1, \tau_1, 1)$ is the amplitude of the probability that there is one electron with momentum p_1 and spin component τ_1, while $\chi_1(p_1, \tau_1, -1)$ is the probability amplitude for the presence of one positron with momentum $-p_1$ and spin component $-\tau_1$. Similarly, $\chi_n(p, \tau, \epsilon)_n$ is the probability of the presence of n_+ electrons and n_- positrons with momenta $\pm p_1, \cdots, \pm p_n$, and spin components $\pm\tau_1, \cdots, \pm\tau_n$, if n_+ of the variables $\epsilon_1, \cdots, \epsilon_n$ have the value $+1$ while n_- of these variables have the value -1. Of course, the functions $\chi_n(p, \tau, \epsilon)_n$ should be *antisymmetric* in $(p_1, \tau_1, \epsilon_1), \cdots, (p_n, \tau_n, \epsilon_n)$.

Annihilation and creation operators of electrons will be denoted by $A^{\mp}(p', \tau', 1)$, those of positrons by $A^{\mp}(p', \tau', -1)$. Note that the time variation of the electron operators is as before

$$A^{\mp}(p', \tau', 1, t) = \exp\{\mp it\omega'\}A^{\mp}(p', \tau', 1); \tag{28.33$_+$}$$

the time variation of the positron annihilation and creation operators

$$A^{\mp}(p', \tau', -1, t) = \exp\{\mp it\omega'\}A^{\mp}(p', \tau', -1), \tag{28.33$_-$}$$

however, agrees with the time variation of the creation and annihilation operators $A_{\cdot 1}^{\pm}$ of electrons with negative energy. For this reason, the positron creation operator $A^+(p', \tau', -1)$ is substituted for the annihilation operator $A_{\cdot 1}^-(p', \tau', -1)$ in the right members of formulas (28.28, .30), while $A^-(p', \tau', 1)$ is substituted for $A_{\cdot 1}^-(p', \tau', 1)$. The resulting expressions are then no longer pure annihilation operators. Customarily, the letter ψ is used to denote these expressions but, since we have used this letter to denote particle representers we shall use the letter Ξ, or rather Ξ^-, for this purpose. Accordingly, we introduce the "field quantities"

$$\Xi^-(x', \sigma', \beta') = \sum_{\epsilon'} R^{p',\tau'}_{x',\sigma',\beta'}(\epsilon')A^{-\epsilon'}(p', \tau', \epsilon') \tag{28.34$^-$}$$

and

$$\Xi^-(x', \sigma', \beta', t) = \sum_{\epsilon'} R^{p',\tau'}_{x',\sigma',\beta'}(\epsilon') \exp\{-it\epsilon'\omega'\}A^{-\epsilon'}(p', \tau', \epsilon'). \tag{28.35$^-$}$$

Here we have used the notation $A^- = A^{-1}$, $A^+ = A^{+1}$ which we find convenient.

The quantity Ξ^- evidently satisfies the same differential equation as $A_{\cdot 1}^-$, see (28.30), namely

$$i\nabla_t\Xi^-(x', \sigma', \beta', t) = H_0^{x',\sigma',\beta'}\Xi^-(x', \sigma', \beta', t). \tag{28.36$^-$}$$

The adjoint Ξ^+ of the operator Ξ^- is given by

$$(28.34)^+ \qquad \Xi^+(x', \sigma', \beta') = \sum_{\epsilon'} \bar{R}^{p:\tau}_{x:\sigma,\beta}(\epsilon')A^{\epsilon'}(p', \tau', \epsilon');$$

its time continuation is governed by the complex conjugate of the differential equation $(28.36)^-$.

A Modified Electron-Positron Field

The process of biquantization can also be carried out with respect to the modified energy operator $H_{mod} = H_0 + V$. A field of modified electrons and positrons can be introduced accordingly. The associated annihilation and creation operators will be denoted by $B^{\mp}(p', \tau', \epsilon')$. Field quantities Ξ^-, Ξ^+ can be introduced by the relation

$$(28.37)^- \qquad \Xi^-(x', \sigma', \beta') = \sum_{\epsilon'} \hat{R}^{p:\tau}_{x:\sigma,\beta}(\epsilon')B^{-\epsilon'}(p', \tau', \epsilon')$$

and its adjoint. The time variation of this operator is then given by formulas similar to (28.33); they satisfy the differential equation

$$(28.38)^- \qquad i\nabla_t\Xi^-(x', \sigma', \beta', t) = H_{mod}^{x:\sigma:\beta}\Xi^-(x', \sigma', \beta', t).$$

Undisturbed annihilation and creation operators $A^{\mp}$ may now again be introduced by the formula

$$(28.39)^- \qquad \Xi^-(x', \sigma', \beta', t) = \sum_{\epsilon'} R^{p:\tau}_{x:\sigma,\beta}(\epsilon')A^{-\epsilon'}(p', \tau', \epsilon', t)$$

and its conjugate; but the time variation of $A^{\mp}$ is no longer given by (28.33).

The *identification* of the field quantity Ξ^- expressed in terms of the modified and unmodified annihilation and creation operators $B^{\mp}$ and $A^{\mp}$ as given by formulas (28.37) and (28.39) furnishes the relationship between the two types of biquantization, that in terms of modified electrons and positrons and that in terms of unmodified ones.

Linear Transformation of Operators A into Operators B

We can eliminate the quantity Ξ^- by applying the inverse of the transformation $\hat{R}^{p:\tau}_{x:\sigma,\beta}$ on both sides of relations $(28.37)^-$ and $(28.39)^-$. In view of formula (28.25) we obtain the relation

$$(28.40)^- \qquad B^{-\epsilon'}(p', \tau', \epsilon') = \sum_{\epsilon''} U^-(\epsilon', \epsilon'')A^{-\epsilon''}(p', \tau', \epsilon'').$$

We recall that each operator $U^-(\epsilon', \epsilon'')$ transforms functions of p', τ' into functions of the same variables. Taking the adjoint of relation $(28.40)^-$ we obtain

$$(28.40)^+ \qquad B^{\epsilon'}(p', \tau', \epsilon') = \sum_{\epsilon''} U^+(\epsilon', \epsilon'')A^{\epsilon''}(p', \tau', \epsilon'').$$

Both relations can be combined into

$$B^{\epsilon}(p', \tau', \epsilon') = \sum_{\epsilon''} U^{\epsilon\epsilon'}(\epsilon', \epsilon'')A^{\epsilon\epsilon'\epsilon''}(p', \tau', \epsilon'')$$

or into

$$\begin{aligned} B^{\epsilon}(p', \tau', \epsilon') &= \sum_{\epsilon''} \tfrac{1}{2}(1 \pm \epsilon'\epsilon'')U^{\epsilon\epsilon'}(\epsilon', \epsilon'')A^{-}(p', \tau', \epsilon'') \\ &\quad + \sum_{\epsilon''} \tfrac{1}{2}(1 \mp \epsilon'\epsilon'')U^{\epsilon\epsilon'}(\epsilon', \epsilon'')A^{+}(p', \tau', \epsilon''). \end{aligned} \tag{28.41}$$

The latter relation is *a linear homogeneous transformation of annihilation and creation operators* of the form $\mathfrak{B} = \mathcal{Y}\mathfrak{A}$ treated in Section 23.

The coefficients $Y_{\bullet\bullet}$ of the pseudo-operator $\mathcal{Y}$ are operators acting on functions of p', τ', ϵ'. We describe them by their matrix elements $Y_{\bullet\bullet}(\epsilon', \epsilon'')$ with respect to the variable ϵ. These matrix elements are then operators acting on functions of p', τ'. We find

$$\begin{aligned} (\epsilon' \mid Y_{--} \mid \epsilon'') &= \tfrac{1}{2}(1 + \epsilon'\epsilon'')U^{-\epsilon'}(\epsilon', \epsilon''), \\ (\epsilon' \mid Y_{++} \mid \epsilon'') &= \tfrac{1}{2}(1 + \epsilon'\epsilon'')U^{\epsilon'}(\epsilon', \epsilon''), \end{aligned} \tag{28.42}$$

$$\begin{aligned} (\epsilon' \mid Y_{-+} \mid \epsilon'') &= \tfrac{1}{2}(1 - \epsilon'\epsilon'')U^{-\epsilon'}(\epsilon', \epsilon''), \\ (\epsilon' \mid Y_{+-} \mid \epsilon'') &= \tfrac{1}{2}(1 - \epsilon'\epsilon'')U^{\epsilon'}(\epsilon', \epsilon''). \end{aligned} \tag{28.43}$$

It is immediately seen that the matrix $\mathcal{Y}$ so given is pseudo-Hermitian. In order to verify that it satisfies the relations $'\mathcal{Y}\mathcal{Y} = 1$ and $\mathcal{Y}'\mathcal{Y} = 1$ we determine the pseudo-operator $'\mathcal{Y}$ from (24.9) and (28.25), (28.26)'.

$$\begin{aligned} (\epsilon' \mid {}'Y_{++} \mid \epsilon'') &= \tfrac{1}{2}(1 + \epsilon'\epsilon'')'U^{\epsilon''}(\epsilon'', \epsilon'), \\ (\epsilon' \mid {}'Y_{--} \mid \epsilon'') &= \tfrac{1}{2}(1 + \epsilon'\epsilon'')'U^{-\epsilon''}(\epsilon'', \epsilon'), \end{aligned} \tag{28.42$'$}$$

$$\begin{aligned} (\epsilon' \mid {}'Y_{-+} \mid \epsilon'') &= \tfrac{1}{2}(1 - \epsilon'\epsilon'')'U^{-\epsilon''}(\epsilon'', \epsilon'), \\ (\epsilon' \mid {}'Y_{+-} \mid \epsilon'') &= \tfrac{1}{2}(1 - \epsilon'\epsilon'')'U^{\epsilon''}(\epsilon'', \epsilon'). \end{aligned} \tag{28.43$'$}$$

Using relation (28.27) we then verify

$$\begin{aligned} \sum_{\epsilon'''} [(\epsilon' \mid {}'Y_{++} \mid \epsilon''')(\epsilon''' \mid Y_{--} \mid \epsilon'') + (\epsilon' \mid {}'Y_{-+} \mid \epsilon''')(\epsilon''' \mid Y_{+-} \mid \epsilon'')] \\ = \delta(\epsilon' - \epsilon''), \end{aligned}$$

$$\sum_{\epsilon'''} [(\epsilon' \mid {}'Y_{++} \mid \epsilon''')(\epsilon''' \mid Y_{-+} \mid \epsilon'') + (\epsilon' \mid {}'Y_{-+} \mid \epsilon''')(\epsilon''' \mid Y_{++} \mid \epsilon'')] = 0$$

and the other relations that constitute the relations $'\mathcal{Y}\mathcal{Y} = \mathcal{Y}'\mathcal{Y} = 1$.

As a consequence of these facts, the theory developed in the preceding sections is applicable.

Modified Vacuum State

Among the various quantities which can be calculated from the elements of the matrices $\mathcal{Y}$ and $'\mathcal{Y}$ we select the operator G which is needed to describe the modified vacuum state

$$\Phi^{B}_{vac} = \eta \exp [-\mathcal{G}]\Phi^{A}_{vac} .$$

Since, as seen from (28.42), the matrix elements of the operator Y_{--} can be written in the form

$$(\epsilon' \mid Y_{--} \mid \epsilon'') = \delta(\epsilon' - \epsilon'')U^{-*'}(\epsilon', \epsilon'), \tag{28.44}$$

the matrix elements of its inverse are

$$(\epsilon' \mid Y_{--}^{-1} \mid \epsilon'') = \delta(\epsilon' - \epsilon'')[U^{-*'}(\epsilon', \epsilon')]^{-1}. \tag{28.45}$$

Since the matrix elements of Y_{-+} can be written as

$$(\epsilon' \mid Y_{-+} \mid \epsilon'') = \delta(\epsilon' + \epsilon'')U^{-*'}(\epsilon', -\epsilon'),$$

see (28.43), those of the operator $G = Y_{--}^{-1} Y_{-+}$ are

$$(\epsilon' \mid G \mid \epsilon'') = \delta(\epsilon' + \epsilon'')[U^{-*'}(\epsilon', \epsilon')]^{-1}U^{-*'}(\epsilon', -\epsilon'). \tag{28.46}$$

It is to be noted that this operator does not depend on the choice of the artificial variables $\hat{p}$, $\hat{\tau}$; for, the operators $U^{-}(\epsilon', \epsilon'')$ transform (p, τ)-representers into $(\hat{p}, \hat{\tau})$-representers and hence the inverse $[U^{-*'}(\epsilon', \epsilon')]^{-1}$ transforms $(\hat{p}, \hat{\tau})$-representers into (p, τ)-representers. The arguments of the kernel

$$(p', \tau', \epsilon' \mid G \mid p'', \tau'', \epsilon'')$$

of the operator G are therefore the original variables associated with the undisturbed field. The antisymmetry of this function is guaranteed by the general theory.

Note that the kernel G vanished unless $\epsilon' = -\epsilon''$. In other words, ***the modified vacuum state consists of pairs of electrons and positrons.*** For example, the amplitude of the probability that there is exactly one such pair of which the first is an electron with momentum p_1 and spin component τ_1, and the second a positron with momentum $-p_2$ and spin component $-\tau_2$ is

$$\sqrt{2}\, \eta(p_1 , \tau_1 , 1 \mid G \mid p_2 , \tau_2 , -1).$$

Vacuum Transition Probability

The probability that the field will be found to be in the unmodified vacuum state if it is known to be in the modified vacuum state—or vice versa—is, by (24.26), (24.45),

$$Pr\,(N, 0 \mid M, 0) = \mid \eta \mid^2 = \exp \{\tfrac{1}{2}\, Tr \log {'Y_{++}} Y_{--}\}.$$

The trace occurring here is easily calculated. From (28.43), (28.42)′ we determine the ϵ-matrix element of the operator $'Y_{++}Y_{--}$ as

$$(\epsilon' \mid 'Y_{++}Y_{--} \mid \epsilon'') = \tfrac{1}{4} \sum_{\epsilon'''} (1 + \epsilon'\epsilon''')(1 + \epsilon'''\epsilon'')'U^{\epsilon'''}(\epsilon''', \epsilon')U^{-\epsilon'''}(\epsilon''', \epsilon'')$$

or, since $1 + \epsilon'\epsilon'' = \delta(\epsilon' - \epsilon'')$,

$$(28.47) \qquad (\epsilon' \mid 'Y_{++}Y_{--} \mid \epsilon'') = \delta(\epsilon' - \epsilon'')'U^{\epsilon'}(\epsilon', \epsilon')U^{-\epsilon'}(\epsilon', \epsilon').$$

From this expression one derives

$$(\epsilon' \mid \log 'Y_{++}Y_{--} \mid \epsilon'') = \delta(\epsilon' - \epsilon'') \log 'U^{\epsilon'}(\epsilon', \epsilon')U^{-\epsilon'}(\epsilon', \epsilon')$$

whence

$$(28.48) \qquad \begin{aligned} Tr \log 'Y_{++}Y_{--} &= \sum_{\epsilon'} Tr \log 'U^{\epsilon'}(\epsilon', \epsilon')U^{-\epsilon'}(\epsilon', \epsilon') \\ &= Tr \log 'U^{+}(1, 1)U^{-}(1, 1) + Tr \log 'U^{-}(-1, -1)U^{+}(-1, -1). \end{aligned}$$

Consequently,

$$(28.49) \qquad \begin{aligned} P(N, 0 \mid M, 0) = \exp \{ \tfrac{1}{2} Tr \log 'U^{+}(1, 1)U^{-}(1, 1) \\ + \tfrac{1}{2} Tr \log 'U^{-}(-1, -1)U^{+}(-1, -1)\}. \end{aligned}$$

Thus *the vacuum transition probability is calculated explicitly* inasmuch as the transformations $U^{\pm}$, $'U^{\pm}$ are supposed to be known.

Note that in the left member of (28.48) one should take the trace of an operator acting on a function of p', τ', ϵ' while in the two right members the traces refer to p', τ' only.

It should also be noted that the probability (28.49) is independent of the choice of the artificial variables $\hat{p}$, $\hat{\tau}$. For, the operators $U^{\pm}(\epsilon', \epsilon'')$ transform (p, τ)-representers into $(\hat{p}, \hat{\tau})$-representers, while the operators $'U^{\pm}(\epsilon', \epsilon'')$ transform $(\hat{p}, \hat{\tau})$-representers into (p, τ)-representers. It follows that the operators $'U^{\pm}U^{\mp}$ transform (p, τ)-representers into (p, τ)-representers and are independent of the choice of the variables $\hat{p}$, $\hat{\tau}$.

It is possible to give the expression (28.49) for the vacuum transition probability a very concise form which involves only the signs $\hat{\epsilon}$ and ϵ of the modified and unmodified energy.

Clearly, as seen from (28.24), (28.27)′, the ϵ-matrix of the operator $\hat{\epsilon}$ is

$$(\epsilon' \mid \hat{\epsilon} \mid \epsilon'') = \sum_{\epsilon'''} 'U^{+}(\epsilon''', \epsilon')\epsilon'''U^{-}(\epsilon''', \epsilon'').$$

The ϵ-matrix of the operator $\tfrac{1}{4}(\hat{\epsilon} - \epsilon)^2$ is therefore

$$\begin{aligned} (\epsilon' \mid \tfrac{1}{4}(\hat{\epsilon} - \epsilon)^2 \mid \epsilon'') &= \tfrac{1}{4}\sum_{\epsilon'''} 'U^{+}(\epsilon''', \epsilon')(\epsilon''' - \epsilon')(\epsilon''' - \epsilon'')U^{-}(\epsilon''', \epsilon'') \\ &= \delta(\epsilon' + \epsilon'')'U^{+}(-\epsilon', \epsilon')U^{-}(-\epsilon', \epsilon') \end{aligned}$$

and consequently its trace is

(28.50) $\frac{1}{2} Tr(\hat{\epsilon} - \epsilon)^2 = Tr\, 'U^+(-1, 1)U^-(-1, 1) + Tr\, 'U^+(1, -1)U^-(1, -1).$

Similarly, we find

$$(\epsilon' \mid 1 - \tfrac{1}{4}(\hat{\epsilon} - \epsilon)^2 \mid \epsilon'') = \delta(\epsilon' + \epsilon'')\, 'U(\epsilon', \epsilon')U^-(\epsilon', \epsilon'),$$

$$(\epsilon' \mid \log [1 - \tfrac{1}{4}(\hat{\epsilon} - \epsilon)^2] \mid \epsilon'') = \delta(\epsilon' + \epsilon'') \log\, 'U^+(\epsilon', \epsilon')U^-(\epsilon', \epsilon')$$

and consequently

$$(28.51)\quad \begin{aligned} &Tr \log [1 - \tfrac{1}{4}(\hat{\epsilon} - \epsilon)^2] \\ &= Tr \log\, 'U^+(1, 1)U^-(1, 1) + Tr \log\, 'U^+(-1, -1)U^-(-1, -1). \end{aligned}$$

Thus we see that *the vacuum transition probability* (28.49) *can be written in the form*

$$(28.52)\quad P(N, 0 \mid M, 0) = \exp \{\tfrac{1}{2} Tr \log [1 - \tfrac{1}{4}(\hat{\epsilon} - \epsilon)^2]\}.$$

The fact that this expression is independent of the quantum variables chosen will prove useful.

If the disturbance is small so that the transformation $\mathcal{Y}$ differs little from the identity one may approximate expression (28.52) by the expression

$$(28.53)\quad P(N, 0 \mid M, 0) \sim 1 - \tfrac{1}{8} Tr(\hat{\epsilon} - \epsilon)^2.$$

In fact, the condition that the trace occurring in (28.52) be finite is equivalent with the condition that the trace $Tr(\hat{\epsilon} - \epsilon)^2$ occurring in (28.53) be finite (cf. the discussion in connection with (25.47), (25.49)).

The right member of (28.53) can be evaluated by (28.50). Employing the kernels $U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'')$ and $'U^+(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'')$ of $U^-(\epsilon', \epsilon'')$ and $'U^+(\epsilon'', \epsilon')$ we find, using (28.26)',

$$(28.53)_1\quad \begin{aligned} P(N, 0 \mid M, 0) \sim 1 - \tfrac{1}{4} \sum_{\tau', \tau''} \iint \{&\mid U^-(p', \tau', 1 \mid p'', \tau'', 1) \mid^2 \\ &+ \mid U^-(p', \tau', -1 \mid p'', \tau'', -1) \mid^2\}\, dp'\, dp''. \end{aligned}$$

Perturbation Approximation

We have supposed the transformations $U^\pm(\epsilon', \epsilon'')$ associated with a single electron to be known. Insofar as this supposition is correct we may say that all relevant quantities such as the number $\mid \eta \mid$ and the operator G can be determined without using an approximation procedure. In most actual cases, however, the determination of the operators $U^\pm(\epsilon', \epsilon'')$ themselves requires the use of an approximation procedure. Under favorable conditions one may employ a perturbation procedure. In typical cases it will mostly be required that the spectrum of the modified energy H_{mod} is the same as that of the unmodified energy H_0 and that, moreover, the artificial variables $\hat{p}$, $\hat{\tau}$, $\hat{\epsilon}$ can be so chosen that the modified absolute energy $\hat{\omega}$ is the same function of $\hat{p}$, $\hat{\tau}$ as the unmodified absolute energy is of p, τ, i.e.,

$$\hat{\omega} = \omega(\hat{p}, \hat{\tau}) = (|\hat{p}|^2 + \mu^2)^{1/2} \tag{28.54}$$

Let $(p', \tau', \epsilon' \mid V \mid p'', \tau'', \epsilon'')$ be the kernel of the disturbing energy and $U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'')$ the kernel of the operator $U^-(\epsilon', \epsilon'')$. Because of (28.54) the latter transformation gives the spectral representation of the operator $H_{mod} = H_0 + V$ and consequently the relation

$$\begin{aligned}\epsilon'\omega' U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'') &= U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'')\epsilon''\omega'' \\ &+ \int \sum_{\tau''', \epsilon'''} U^-(p', \tau', \epsilon' \mid p''', \tau''', \epsilon''') \\ &\cdot (p''', \tau''', \epsilon''' \mid V \mid p', \tau', \epsilon')\, dp'''\end{aligned} \tag{28.55}$$

holds. Since

$$U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'') = \delta(p' - p'', \tau' - \tau'', \epsilon' - \epsilon'') \tag{28.56}$$

in first approximation, equation (28.52) becomes in first approximation

$$\begin{aligned}&\epsilon'\omega' U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'') \\ &= U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'')\epsilon''\omega'' + (p', \tau', \epsilon' \mid V \mid p'', \tau'', \epsilon'').\end{aligned} \tag{28.57}$$

A solution of this equation is

$$\begin{aligned}U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'') &= \delta(p' - p'', \tau' - \tau'', \epsilon' - \epsilon'') \\ &+ [(\epsilon'\omega' - \epsilon''\omega'')^{-1} + i\pi\,\delta(\epsilon'\omega' - \epsilon''\omega'')](p', \tau', \epsilon' \mid V \mid p'', \tau'', \epsilon'')\end{aligned} \tag{28.58}$$

For $\epsilon'' = -\epsilon'$, in particular,

$$\begin{aligned}&U^-(p', \tau', \epsilon' \mid p'', \tau'', -\epsilon') \\ &= \epsilon'(\omega' + \omega'')^{-1}(p', \tau', \epsilon' \mid V \mid p'', \tau'', -\epsilon').\end{aligned} \tag{28.59}$$

Instead of the factor $i\pi$ the factor $-i\pi$ could just as well have been chosen on the right hand side of (28.58). In either case the operator U is unitary in first approximation, cf. [2] and [44]. However, the sign of this factor is irrelevant in the present context since it does not contribute to the right member of (28.59).

The expression $Tr\ (\hat{\epsilon} - \epsilon)^2$ which occurs in the approximate expression

(28.53) for the vacuum transition probability can now, by (28.54), be approximately expressed as

$$1 - Pr\,(N, 0 \mid M, 0) \sim \tfrac{1}{2} \sum_{\tau', \tau''} \iint \{ \mid (p', \tau', 1 \mid V \mid p'', \tau'', -1) \mid^2 \tag{28.60}$$

$$+ \mid (p', \tau', -1 \mid V \mid p'', \tau'', 1) \mid^2 \} (\omega' + \omega'')^{-2}\, dp'\, dp''.$$

This formula was found by H. E. Moses [51] who applied a perturbation procedure directly to the field equations. In the following we present a slight modification of his arguments.

It is possible to write expression (28.60) in a form which does not depend on the choice of τ, ϵ or σ_3, β as quantum variables. To this end we introduce the operator

$$W = \tfrac{1}{4}(1 + \epsilon)V(1 - \epsilon) + \tfrac{1}{4}(1 - \epsilon)V(1 + \epsilon) = \tfrac{1}{2}V - \tfrac{1}{2}\epsilon V \epsilon. \tag{28.61}$$

Its (p, τ, ϵ)-kernel is

$$(p', \tau', \epsilon' \mid W \mid p'', \tau'', \epsilon'') = \delta(\epsilon' + \epsilon'')(p', \tau', \epsilon' \mid V \mid p'', \tau'', \epsilon''), \tag{28.62}$$

and hence we have

$$1 - Pr\,(N, 0 \mid M, 0) \tag{28.63}$$
$$\sim \frac{1}{2} \iint \sum_{\tau', \tau''} \sum_{\epsilon', \epsilon''} \mid (p', \tau', \epsilon' \mid W \mid p'', \tau'', \epsilon'') \mid^2 (\omega' + \omega'')^{-2}\, dp'\, dp''.$$

Next we assign to every operator A acting on states Ψ of single particles its "p-kernel"; i.e. the operator $(p' \mid A \mid p'')$ which acts on "spin states" represented by functions of σ_3, β or τ, ϵ and whose (p, τ, ϵ)- and (p, σ_3, β)-representers are $(p', \tau', \epsilon' \mid A \mid p'', \tau'', \epsilon'')$ and $(p', \sigma_3', \beta' \mid A \mid p'', \sigma_3'', \beta'')$ respectively. The operator with the kernel $\overline{(p'', \tau'', \epsilon'' \mid A \mid p', \tau', \epsilon')}$ will be denoted by $(p' \mid A^* \mid p'')$.

Observing that the (p, τ, ϵ)-representer of the operator $(p'' \mid W^* \mid p') \cdot (p' \mid W \mid p'')$ is $\sum_{\tau'''} \sum_{\epsilon'''} \overline{(p', \tau''', \epsilon''' \mid W \mid p'', \tau', \epsilon')}(p', \tau''', \epsilon''' \mid W \mid p'', \tau'', \epsilon'')$ we see that we may write

$$\sum_{\tau' \tau''} \sum_{\epsilon' \epsilon''} \mid (p', \tau', \epsilon' \mid W \mid p'', \tau'', \epsilon'') \mid^2 = Tr\,(p'' \mid W^* \mid p')(p' \mid W \mid p'')$$

and hence formula (28.63) becomes

$$1 - P(N, 0 \mid M, 0) \tag{28.64}$$
$$\sim \frac{1}{2} \iint Tr\,(p'' \mid W^* \mid p')(p' \mid W \mid p'')(\omega' + \omega'')^{-2}\, dp'\, dp''.$$

The trace occurring here refers to the variables τ, ϵ; but, being independent of the choice of the variables, it could just as well be evaluated by employing the variables σ_3, β.

This evaluation will be facilitated by using the operators $(p' \mid \epsilon \mid p'')$. From the definition (28.13) of ϵ and Ω, and the definition (28.3) of H_0 we see that the operator $(p' \mid \epsilon \mid p'')$ can be written in the form

$$(p' \mid \epsilon \mid p'') = \delta(p' - p'')\epsilon(p') \tag{28.65}$$

where $\epsilon(p')$ is an operator acting on spin states and depending on p'; specifically

$$\epsilon(p') = [\omega']^{-1}[\beta_1(\sigma p') + \mu\beta]. \tag{28.66}$$

Using this operator we may derive from (28.61) the expression

$$2(p' \mid W \mid p'') = (p' \mid V \mid p'') - \epsilon(p')(p' \mid V \mid p'')\epsilon(p''). \tag{28.67}$$

Since $[\epsilon(p')]^2 = 1$, we have

$$\begin{aligned}4(p'' \mid W^* \mid p')(p' \mid W \mid p'') &= (p'' \mid V^* \mid p')(p' \mid V \mid p'')\\ &- \epsilon(p'')(p'' \mid V^* \mid p')\epsilon(p')(p' \mid V \mid p'') - (p'' \mid V^* \mid p')\epsilon(p')(p' \mid V \mid p'')\epsilon(p'')\\ &+ \epsilon(p'')(p'' \mid V^* \mid p')(p' \mid V \mid p'')\epsilon(p'')\end{aligned}$$

whence, since $Tr\, AB = Tr\, BA$, the formula

$$\begin{aligned}&2\, Tr\, (p'' \mid W^* \mid p')(p' \mid W \mid p'')\\ &= Tr\, (p'' \mid V^* \mid p')(p' \mid V \mid p'') - Tr\, \epsilon(p'')(p'' \mid V^* \mid p')\epsilon(p')(p' \mid V \mid p'')\end{aligned} \tag{28.68}$$

results; it may be used in (28.64).

Expression (28.68) can be greatly simplified in the important case in which the operator V is constant as regards spin states so that $V^{\nu,\sigma_3,\beta} = V^{\nu}$ acts on functions $\psi(p, \sigma_3, \beta)$ only insofar as they depend on p. In this case the operator $\epsilon(p')$ commutes with the operator $(p' \mid V \mid p'')$ and hence we have

$$\begin{aligned}&2\, Tr\, (p'' \mid W^* \mid p')(p' \mid W \mid p'')\\ &= [Tr\, 1 - Tr\, \epsilon(p'')\epsilon(p')](p'' \mid V^* \mid p')(p' \mid V \mid p'').\end{aligned} \tag{28.69}$$

The trace of the identity as regards spin states is evidently

$$Tr\, 1 = 4. \tag{28.70}$$

The trace of $\epsilon(p'')\epsilon(p')$ is easily evaluated from (28.66) combined with the commutation relations (28.1),

$$Tr\, \epsilon(p'')\epsilon(p') = 4(\omega'\omega'')^{-1}[p''p' + \mu^2]. \tag{28.71}$$

Hence

$$\tfrac{1}{4} Tr\,[1 - \epsilon(p'')\epsilon(p')] = 1 - (\omega'\omega'')^{-1}(p'p'' + \mu^2)$$
$$= (\omega'\omega'')^{-1}[\omega'\omega'' - p'p'' - \mu^2]$$

and formula (28.64) becomes

$$(28.72)\quad \begin{aligned} &1 - P(N, 0 \mid M, 0) \\ &= \iint \mid (p' \mid V \mid p'') \mid^2 [\omega'\omega'' - p'p'' - \mu^2](\omega'\omega'')^{-1}(\omega' + \omega'')^{-2}\, dp'\, dp''. \end{aligned}$$

Of particular interest is the special case where the disturbing energy V is an external potential which is a function of the position x. Then the p-kernel of V is a function of $p' - p''$,

$$(p' \mid V \mid p'') = V(p'' - p').$$

Moses observed [49] that in this case the right member of formula (28.72) is finite provided the function $V(p)$ dies out sufficiently strongly as $\mid p \mid \to \infty$. In fact, it is sufficient that $\mid V(p) \mid$ be bounded and integrable. To see this we may introduce $\tilde{p} = p'' - p'$ and p' as new variables and set

$$(28.73)\quad K(p') = (\omega')^4 \int \mid V(\tilde{p}) \mid^2 [\omega'\omega'' - p'p'' - \mu^2](\omega'\omega'')^{-1}(\omega' + \omega'')^{-2}\, d\tilde{p},$$

so that

$$(28.74)\quad 1 - P(N, 0 \mid M, 0) = \int K(p')(\omega')^{-4}\, dp'.$$

For large values of $\mid p' \mid$ we have

$$\omega'\omega'' - p'p'' - \mu^2 \sim \tfrac{1}{2} \mid p' \mid^{-2} [\mid p' \mid^2 \mid \tilde{p} \mid^2 - (p'\tilde{p})^2].$$

Here the right member remains bounded and the same is, of course, true of $(\omega')^2(\omega'')^{-1}(\omega' + \omega'')^{-2}$. Consequently, $K(p')$ is bounded for large values of $\mid p' \mid$ and hence the integral in (28.74) is finite.

This integral gives only the contribution of second order to the probability $P(N, 0 \mid M, 0)$. A closer analysis would show that the complete probability itself is also finite *provided the disturbance V is small enough*. In this case then *the canonical transformation T exists*. It is true that the expectation values of certain observables—such as the charge density at any point—are infinite in the vacuum state. One must expect such occurrences whenever the operators which correspond to the observables are unbounded. Of course, such occurrences alone do not invalidate the theory.

The situation is quite different if the disturbance V is not constant as regards spin states, for example in case

$$V = (\alpha A) = \beta_1(\sigma A)$$

where $A = \{A_1, A_2, A_3\}$ is the vector potential of an external electromagnetic field. In this case relation (28.68) leads to

$$\tfrac{1}{2}\, Tr\,(p'' \mid W^* \mid p')(p' \mid W \mid p'') = (\omega'\omega'')^{-1}[\omega'\omega'' + p'p'' + \mu^2]$$

provided A is divergence-free, pA = 0, which we may assume. The arguments that led to the boundedness of the expression $K(p')$ now break down and, consequently, the right member of formula (28.34) is infinite. It would seem that one should assume the field to be myriotic in this case and it may perhaps be that one could then rederive in a mathematically satisfactory way some of the finite results that have been obtained by using—in intermediate steps—certain terms which are actually infinite.

Bibliography to Part V

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